APPLICATION OF A DIRECT VARIATIONAL PRINCIPLE IN ELASTIC STABILITY ANALYSIS OF THIN RECTANGULAR FLAT PLATES

BY

IBEARUGBULEM, OWUS MATHIAS (B. ENG., M. ENG.)

20064719058

BEING
A RESEARCH PROJECT SUBMITTED TO THE

POSTGRADUATE SCHOOL

FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI

IN PARTIAL FULFILMENT OF THE

REQUIREMENTS FOR THE AWARD OF DEGREE OF

DOCTOR OF PHILOSOPHY (Ph. D)

IN

STRUCTURAL ENGINEERING

MAY, 2012

CERTIFICATION

I, Owus Mathias Ibearugbulem (20064719058), hereby certify that this research project, "Application of a Direct Variational Principle in Elastic Stability of Thin Rectangular Flat Plates" is original to me and has not been submitted elsewhere for the award of any degree.

Signature----------------------------- Date ----------------

Owus Mathias Ibearugbulem (20064719058)

APPROVAL

This research project, "Application of a Direct Variational Principle in Elastic Stability of Thin Rectangular Flat Plates" by Ibearugbulem, O. M. is hereby approved as a satisfactory research project submitted for the award of degree of Doctor of Philosophy (Ph. D) in Structural Engineering.

Signature---------------------------- Date ----------------
Engr. Rev. Prof. N. N. Osadebe
Supervisor,

Signature---------------------------- Date ----------------
Engr. Dr. D. O. Onwuka
Co-Supervisor

Signature---------------------------- Date ----------------
Engr. Dr. J. C. Ezeh
Co-Supervisor

Signature---------------------------- Date ----------------
Engr. Dr. J. C. Ezeh
Head of Department

Signature---------------------------- Date ----------------
Engr. Prof. E. E. Anyanwu
Dean of School of Engineering and Engineering Technology

Signature---------------------------- Date ----------------
Engr. Prof. C. D. Okereke
Dean of Postgraduate School

Signature---------------------------- Date ----------------
Engr. Prof. O. E. Alutu
External examiner

DEDICATION

This project report is dedicated to my sweet heart (my wife), Mrs. Uchechi M. Ibearugbulem, my son Amarachi Ibearugbulem, my daughters Adaugo Ibearugbulem and Akarachinyerem Ibearugbulem, and my parents Mr. Aloysius I. Uzoigwe & Mrs. Adaugo F. Uzoigwe.

ACKNOWLEDGEMET

My profound gratitude goes to my supervisor, Engr. Rev. Prof. Nkemakolam Nwaolisa Osadebe for initiating and supervising this research project. May God bless him. I am also grateful to my second supervisor; Engr. Dr. Davies O. Onwuka for his indispensible guidance. He made invaluable contribution to the execution and report of the research. God will bless him. Engr. Dr. John C. Ezeh, my third supervisor stood by me and made sure I got the necessary assistance I needed during the research. He provided me with literature materials and followed the research at every stage of the work. May he receive the blessings of God. The entire staff and management Civil Engineering Department, Federal University of Technology, Owerri (FUTO) will remain blessed. They contributed in one way or the other to the success of this research. They gave me enabling environment to work in. My gratitude goes to the staff and management of FUTO library for their assistance in availing literature and internet materials to me. I wish to express my appreciation to my elder brother, Rev. Fr. Chinedu Ibearugbulem for his moral support. He also provided me with some internet facilities that help me very well in this research. I will not forget my wife Mrs. Uchechi M. Ibearugbulem who stood by me during the research. She made home environment conducive for this research. Finally, I want to thank Mr. Ogedi H. Ibearugbulem, my brother, and the entire staff and management of IBE'S Computer Institute, 190 Wetheral Road Owerri for type setting, printing and binding this project.

Above all, I am very grateful to God Almighty who made this research a success. He provided live, energy, wisdom and enabling environment for this work. His grace on me and all the people that contributed to this research made this research a success. God may Your Name be held Holy and may You be praised, adored, glorified and worshiped for ever and ever. Amen

Engr. Owus Mathias Ibearugbulem

ABSTRACT

The purpose of this work is to use the energy approach in the form of direct variational principle (Rayleigh-Ritz method) for buckling analysis of thin rectangular plates with various boundary conditions using Taylor series shape functions. To do this, thin rectangular flat plate of various boundary conditions with three dimensions Ly, Lx and t was analyzed in this research. Ly and Lx are secondary and primary in-plane dimensions respectively and t is the plate thickness. The boundary conditions covered in this research included SSSS, SSSC, SSCC, SCCC, SCSC, CCCC, SSSF, CCCF and SCFS plates. Rayleigh-Ritz method of direct variational approach for the plate analysis was adopted. The total potential energy functional of the method was derived from first principle by using equations and principles of theory of elasticity. Taylor-MacLaurin's series was used to formulate the approximate shape functions for the plate with various boundary conditions. The shape functions from Taylor-MacLaurin's series were substituted into the total potential energy functional, which was subsequently minimized to get the stability equations. Derived Eigen-value solver was used to solve the stability equations for plates of various aspect ratios (from 0.1 to 1 at the increment of 0.1) to get the buckling loads of the plates. The buckling loads from this study were compared with those of earlier researches. The results showed that the average percentage differences recorded for SSSS, CCCC, CSCS, CSSS, and SSFS plates are 0.069%, 3.54%, 3.071%, 6.25% and 4.14% respectively. The convergence of the shape function showed that for CSCS plate, the difference between the buckling loads when the Taylor-MacLaurin's series were truncated at m = n = 4 and m = n = 5 is 1.11%. This difference is 0.878% for CCSS plates. These differences showed that the shape functions formulated by using Taylor-McLauruin's series has rapid convergence and very good approximation of the exact displacement functions of the deformed thin rectangular plate under in-plane loading.

TABLE OF CONTENT

LIST OF FIGURES

LIST OF TABLE

DEFINITION OF NOTATIONS

NOTATIONS	MEANING
X	The primary axis of the plate. That is the shorter of the two axes of the major plane of the plate
Y	The secondary axis of the plate. That is the longer of the two axes of the major plane of the plate
Z	The tertiary axis of the plate. That is the shortest of the three axes of the plate
a	Length of the primary dimension of the plate
b	Length of the secondary dimension of the plate
t; h	Thickness of the plate or the length of the tertiary dimension of the plate
P	Is the aspect ratio. That is P = a / b
M	The number of terms in the truncated series with respect to x direction
N	The number of terms in the truncated series with respect to y direction
S	Simple support
C	Clamped of fixed support
F	Free of support
SSSS	Four edges of plate are of simple support
CCCC	Four edges of the plate are clamped
SSCC	Two adjacent edge of the plate are of simple support and the other two are clamped
SSSC	Three edges of the plate are of simple support and the last edge is clamped
SCCC	The first of the plate is of simple support and the rest are clamped
SCSCS	Two opposite edges of the plate are of simple support and the other opposite edges are clamped
SCSF	Two opposite edges of the plate are of simple support, second and last edges are clamped and free of support respectively
CFSS	First and second edges of the plate are clamped and of free support respectively, while the other edges are of simple support
CFCC	Three edges of the plate are clamped and the remaining is free of support

$(Nx)_{cr}$	Critical buckling load in x direction
$(Ny)_{cr}$	Critical buckling load in y direction
u	Displacement of the plate in x direction
v	Displacement of the plate in y direction
W =W(x,y)	Plate displacement in z direction. It is a function of x and y.
Σ	stress
Є	strain
E	Modulus of elasticity
e	Material density
G	Acceleration due to gravity
Δ	Change in length of the continuum
EW	External work
U	Internal (strain) energy
G	Shear modulus of the plate
T	Shear stress of the plate
Γ	Shear strain of the plate
μ	Poisson's ratio
D	Modulus of flexural rigidity of the plate
π; Π	Total potential energy of the plate
R	Non dimensional axis (quantity) parallel to x axis. R = x/a
Q	Non dimensional axis (quantity) parallel to y axis. R = y/a
w^{IX}	First partial derivative of W with respect to x. $w^{Ix} = w(x,y)^{Ix} = \frac{\partial w(x,y)}{\partial x}$
w^{IY}	First partial derivative of W with respect to y $w^{Iy} = w(x,y)^{Iy} = \frac{\partial w(x,y)}{\partial y}$
w^{IIX}	second partial derivative of W with respect to x. $w^{IIx} = w(x,y)^{IIx} = \frac{\partial^2 w(x,y)}{\partial x^2}$

w^{IIY}	Second partial derivative of W with respect to y
	$w^{IIy} = w(x,y)^{IIy} = \frac{\partial^2 w(x,y)}{\partial y^2}$
w^{IIXY}	Second partial derivative of W with respect to both x and y
	$w^{IIxy} = w(x,y)^{IIxy} = \frac{\partial^2 w(x,y)}{\partial x \partial y}$
w^{IR}	$a.w^{IX}$
w^{IQ}	$b.w^{Iy}$
w^{IIR}	$a^2.w^{IIX}$
w^{IIQ}	$b^2.w^{IIy}$
w^{IIRQ}	$a.b.w^{IIxy}$
N_x	Buckling (in plane)load in x direction
N_y	Buckling (in plane)load in y direction
∂R	$\partial R = \partial x / a$
∂Q	$\partial Q = \partial y / b$

CHAPTER ONE

INTRODUCTION

1.1 BACKGROUND OF STUDY

Three approaches are used in the solution of elastic stability analysis of thin plates. They are the equilibrium (Euler) approach, the energy (approximate) approach and the numerical approach. The Euler approach tends to find solution of the governing differential equation by direct integration and satisfying the boundary conditions of the four edges of the plate. The rectangular plate has four edges and the numbering of the edges is shown in figure 1:

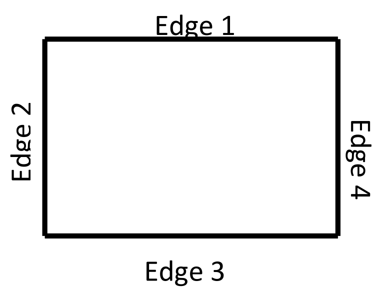

Figure 1.1: Rectangular plate with edge numbering

Some of the boundary conditions of the edges of a rectangular plate are:

S designates simply support

C designates clamped support

F designates free support

A rectangular plate is unique from the other by the conditions of its four edges. SCFS for instance means that edges 1and 4 are simply supported, edge 2 is clamped and edge 3 is free of support. Various rectangular plates, which are distinct from one another, are shown in figure 1.2.

Direct integration leads to stability equation, from which solution is obtained. This solution is called the exact solution. Unfortunately, the method can only be used to analyze plates that are simply supported along the four edges, which is SSSS plate. When one of the edges is not simply supported, the approach becomes very tedious and difficult. In other words, it can be extremely difficult to analyze a plate that has at least one of its edges not simply supported along the four edges. Consequently, the use of other approaches has become very necessary and imperative.

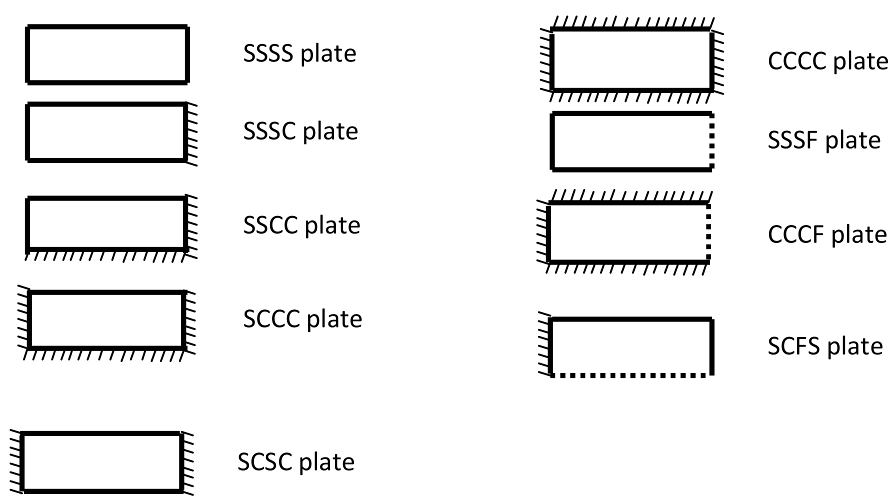

Figure 1.2: Plates of various boundary conditions

Numerical approach is a good alternative to the Euler approach. Some examples of this approach include truncated double Fourier series, finite difference, finite strip, Runge-Kutta and finite element methods among others. Methods of numerical approaches have the capacity of handling plates of various boundary conditions. It has been shown from past works that in most cases, the solution from numerical approach approximate closely those of the exact approach (Ventsel and Krauthammer, 2001). The problem with these numerical solutions is that the accuracy of the solution is dependent on the amount of work to be done. For instance, if one is using finite element method, the more the number of elements used in the analysis the closer the approximate solution to the exact solution. Hence, when a plate has to be divided into several elemental plates for an accurate solution to be reached, then the extensive analysis is involved, requiring enormous time to be invested. Outside the time input, extensive analysis and the great volume of data generated will be difficult to condense into design charts and tables. Although, the data storage capacity of a microcomputer is high, the volume may exceed the memory of a particular computer, with the result that the computer goes out of memory. In view of this, band width control becomes necessary in matrix formulation. A sound knowledge in mathematics and skilful experience in computer programming is inevitable in this case. At this point one will see vividly that the problem one is trying to avoid in equilibrium approach is still found in numerical approach.

Energy approach is another method that can be used. This approach is quite different from Euler and numerical approaches. The solution from it agrees approximately with the exact solution. Typical examples of energy approaches are Ritz, Raleigh-Ritz; Garlekin, minimum potential energy etc. These methods are called variational methods. They seek to minimize the total potential energy functional in order to get the stability matrix. This functional is a function of plate deflection function. The accuracy of the solution is dependent on the accuracy of the approximate deflection function (shape function). Approximate shape function is substituted in the total potential energy functional, and the resulting equation is partially differentiated. The total potential energy will be said to be minimized when its partial derivative is equated to zero. This implies that the difference between the approximate and exact solutions is zero (Iyengar, 1988).

The more the approximate shape function gets closer to the exact shape function the more the approximate solution gets closer to the exact solution. Many scholars have used trigonometric series in this approach. For instance, trigonometric series can be used to formulate approximate shape function for a plate, whose four edges are simply supported or clamped. It can also be used for a plate whose opposite edges are clamped and the other opposite edges are simply supported. However, it is extremely difficult to formulate a shape function for plates using trigonometric series when opposite edges are clamped and simply supported like propped cantilever beams. Examples of plates, whose shape function can not be formulated using trigonometric series, include SSCC plate, SCCC plate, CSSS plate, SSCF plate, CSSF plate etc. Some other boundary conditions make it difficult to use the trigonometric series (Ugural, 1999, Iyengar, 1988, Ventsel and Krauthammer, 2001).

Because of these limitations of energy method using trigonometric series to formulate the approximate shape functions, one will be tempted to use the numerical approach. In the light of the above problems, researches in thin plate buckling are going on so as to obtain solutions that are very close approximation to exact solution, and at the same time reduce the volume of computation.

Consequently, this research sought to formulate shape functions using Taylor series. For different cases with different boundary conditions, Taylor series are used to obtain approximate shape functions. The resulting approximate shape functions are substituted into the total potential energy functional which is then minimized to get the stability equations.

The stability equations are, in most cases, Eigen-value matrix equations involving consistent mass as against lumped mass. Polynomial method of Eigen-value problem is the only method from literature review that can effectively handle Eigen-value equations containing consistent mass. However, solution using Polynomial method becomes intractable when the size of the matrix is up to 4 x 4.

It is in attempt to address these problems that gave birth to the research topic "Application of a Direct Variational Principle in Elastic Stability Analysis of Thin Rectangular Flat Plates".

1.2 STATEMENT OF PROBLEM

The problems of this research are stated as:

i. Direct integration is somewhat easy only for the condition of SSSS plate. It is very difficult for any other plate that is not SSSS plate. Direct integration results in exact solutions.

ii. Previous research works had formulated shape functions by using trigonometric functions in energy approach. The problem with trigonometric functions is that they can not be used to formulate shape functions for CSSS plate, CCSC plate, and CCSS plate.

iii. In the light of the above problems, numerical approaches were inevitable. The problem of the numerical methods is the amount of work involved in the formulation, and expertise in the use of computer. The accuracy of the numerical methods depends on the number of finite units the plate is divided into.

iv. Wit

h all these problems, the present research will use energy approach in the form of a direct variational principle (Rayleigh-Ritz Method). The approximate shape function will be formulated by using Taylor series as against using trigonometric functions.

1.3 OBJECTIVE OF STUDY

The objectives of this proposed research include the following:

i. To use Taylor series in Raleigh-Ritz method for buckling analysis of thin rectangular flat plate.
ii. To develop a new Eigen-value solver that can handle consistent matrix form of Eigen-value.
iii. To find how close the obtained solutions of plate buckling analysis using Taylor-MacLaurin's series are to exact solutions.

1.4 JUSTIFICATION/CONTRIBUTION OF THE STUDY

This research has

i. Contributed in addressing problem of dearth of literature in the use Taylor series approximation of shape functions.

ii. Contributed in addressing problem of dearth of literature for eigenvalue solvers to effectively handle consistent mass eigenvalue problem of a matrix size of up to 4 x 4 and above.

iii. Exposed the potentials of the use of Taylor series as against trigonometric series in analysis of plates and shells problems.

iv. Provided Solution for CSSS plate, CCSC plate, CCSS plate and CCFC plate using Raleigh-Ritz method and Taylor series.

1.5 SCOPE OF STUDY

A flat rectangular thin plate has three dimensions Ly, Lx and t. Where Ly and Lx are respectively secondary and primary in-plane dimensions and t is the plate thickness. Lx/t is used to classify a plate as thick, stiff, thin or membrane. If the ratio, Lx/t is less than ten (10) then the plate is thick. If the range, $10 \leq Lx/t \leq 100$ holds then the plate is thin. If the ratio, Lx/t is greater than hundred (100) then the plate is a membrane. This research studied thin plates. A plate can be flat (of uniform thickness) or of varying thickness. The plate can also be rectangular, circular or any other polygonal shape. However, this research was concerned with rectangular shape. The various boundary conditions covered in this research included SSSS, SSSC, SSCC, SCCC, SCSC, CCCC, SSSF, CCCF and SCFS plates

After defining the types of plate analyzed in this study, the stages involved in the course of executing the project were stated as thus. Literature review and internet search were made in this proposed research. This enabled the proposed research to discover some unanswered theoretical questions, and know the extent past scholars had gone in this direction. This was to avoid repetition of a study that had been made before now. Chapter two under the title of "**literature review**" handled this. The literature review shall be followed by the formulation of **total potential energy functional** from the first principle by using the equations and principles **theory of elasticity**. The substitution of various boundary conditions as concerned various cases of rectangular thin plates into the Taylor series to approximate various shape functions followed. These shape functions were substituted into the formulated **total potential energy functional**. The resulting **total potential energy functionals** for various plates of different boundary conditions were minimized. This minimization gave the stability equations for various plates. All of these were done in chapter three under the title of "**method**". Solution of various plates by substituting the various **aspect ratios** (from 0.1 to 1 at the increment of 0.1) into the stability equations was the next thing that was done. The resultant solutions were compared with the solutions from the use of trigonometric series where available. Comparison was also made with some known exact solutions. Some available results from numerical methods were used to compare with the solutions of this research. General discussions and comments were made as concerned the comparisons. All these were done in chapter four under the title "**results**". The next thing that was done is drawing of conclusions based on the results and making sundry recommendations. This was treated in chapter five under the title of "**conclusion**".

CHAPTER TWO
LITERATURE REVIEW

2.1 HISTORY OF THIN PLATES

Some of the histories of plate as documented in this research were attributed to the works of Szilard (2004) and ventsel (2001). The history goes this way. The first published work related to thin plates analysis was made by Bernoulli (1789). This work involved the flexural analysis of elastic elements, which gave rise to the flexural theory in elasticity. This is known today as Bernoulli flexural theory. However, Bernoulli was not the first person in the history. A great mathematician called Euler (who lived between 1707 and 1783) was actually the first person who worked on thin elements in elasticity. This is popularly known as Euler membrane theory. Navier (1823) also worked on flexural analysis of elastic elements. The works of Bernoulli and Navier laid the foundation of various studies in thin plates. Kirchhoff (1877) and Venant (1883) in their separate studies worked on the theory for combined membrane and flexural effects on thin elements and gave a governing equation for an isotropic plate that is loaded with in–plane loads of Nx, Ny and Nxy and also loaded with a transverse load of p as:

$$P + Nx\frac{\partial^2 W}{\partial X^2} + 2Nxy\frac{\partial^2 W}{\partial X\,\partial Y} + Ny\frac{\partial^2 W}{\partial Y^2} \quad D\left[\frac{\partial^4 W}{\partial X^4} + 2\frac{\partial^4 W}{\partial X^2\,\partial Y^2} + \frac{\partial^4 W}{\partial Y^4}\right] = 0 \quad (2.1)$$

The unit of the in-plane load is N/m while that for the transverse load is N/ m^2. This equation is similar to equation 217 of Timoshenko and Krieger (1970). Later it was discovered that the governing equation has some limitations. This was because it was noticed that the in-plane forces vary as the plate buckles. In order to correct thin deficiency, Kerman (1910) introduced a stress function ϕ to modify the previous governing equation and came up with another governing equation as:

$$P + \frac{\partial^4 \phi}{\partial x\, \partial y} + 2\frac{\partial^4 \phi}{\partial x^2\, \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} + \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \quad 2\frac{\partial^2 \phi}{\partial x\, \partial y}\frac{\partial^2 w}{\partial x\, \partial y}$$

$$= Et\left[\left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] = D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\, \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] \qquad (2.2)$$

With this equation one can easily study or analyze large deflection of thin plates. In such analysis the in-plane forces would be taken as zero.

Huber (1914) studied the behaviours of orthotropic plates and developed a governing equation (equation 2.3) for a transverse load of p and used it to analyze a reinforced concrete slab. The governing equation is:

$$P - D_x\frac{\partial^4 w}{\partial x^4} - 2H\frac{\partial^4 w}{\partial x^2\, \partial y^2} - D_y\frac{\partial^4 w}{\partial y^4} = 0 \qquad (2.3)$$

Energy principle of flat plate analysis was used by Timoshenko (1907) to solve the problem of a simply supported rectangular plate with two opposite sides carrying uniform compressive loads. The energy principles was used in analyzing flat plates with various boundary conditions (Timoshenko, 1910 and 1913). Timoshenko (1921) also used energy principle to study the buckling of thin plates under the shear forces uniformly distributed along the plate sides. In this an approximate solution for the critical shear force of a rectangular flat plate that is simply supported was got. This approximate solution was in use until Bergmann and Reissner (1932) developed an exact solution for critical shear forces for buckling of flat plates that are simply supported. The difference between the exact solution and the approximate solution was about 5% - 7%. Seydel (1933) and Stein and Neff (1947) on their own, separately developed exact solutions for critical shear forces for buckling of flat plates that were simply supported. Stein and Neff (1947) also proved that the symmetric buckling mode was higher than that of antisymmetirc buckling mode over a narrow range of aspect ratios. Other scholars like Bollenrath (1931), Cox (1933) and Moheit (1942) analysed flat rectangular plates of various edge supports.

Reissner (1945) studied the effect of shear deformation on critical buckling force and discovered that the effect was high enough to be ignored. In the light of this, a solution that takes into consideration, the shear deformation; in determining the accurate buckling load was developed. Reissener`s solution was worked upon by Minidlin (1951). Mindlin came up with a solution that handles flat plate vibration. It was discovered that these solutions were becoming complex as the deflection was increasing to the order of the plate thickness and more. In this case, the solutions were no longer linear. Marguerre (1938) and Way (1938) in separate works tried to solve this problem by using the energy principle in getting solutions that addressed the problem. In their own separate works, Levy (1942) and Levy and Greeman (1942) employed truncated double Fourier series in getting solutions that also addressed the earlier problem. The advent of micro computers in the early 1960`s paved way for numeric techniques. Some of these techniques included the finite difference method, finite strip method, finite element method and relaxation techniques etc. Basu and Chapman (1966) used finite difference method to solve isolated flat plate problems with the aid of computer. Fok (1980) also used finite difference method in obtaining a satisfactory solution of analysis of imperfect box-Column and imperfect channel- column.

2.2 PREVIOUS WORKS ON THIN PLATES

Azhari and Bradford (1994) tried to calculate the elastic local buckling stress of plate assemblies. In doing this, they augmented the bubble functions to the ordinary semi-analytical complex finite strip treatment. Bubble functions were finite element shapes that were zero on the boundary of the element, but nonzero at the other points. Their result showed that the use of bubble functions improved significantly the convergence of finite strip method in terms of strip subdivision. This led to much smaller storage required for the structural stiffness and stability matrices.

Ye (1994) studied the nonlinear behavior of rectangular thin plates with initial imperfections. In this work, Iterative boundary element and finite element methods were employed. Spline function was used in both the iterative element and finite element methods. The imperfections were expressed by double Fourier series.

Buckling of rectangular plates under uniaxial loading was studied by Plaut and Guran (1994). They used plates and loads that are uniform. The loaded edges were restrained in position and direction while the unloaded edges were simply supported. They used Bulson (1970) governing equation. This equilibrium equation was solved analytically. In doing this they arrived at two characteristic equations. One of the equations corresponds to buckling modes that are symmetric in the direction of loading. The other equation corresponds to anti-symmetric modes. Their result showed that the critical load of a rectangular plate might be increased significantly if the loaded edges stiffen as they were compressed.

Rao, Naidu and Raju (1994) studied the post buckling of moderately thick circular plates with edge elastic restraint in their work, they studied both the mechanical and thermal post buckling behaviors of elastically restrained and moderately thick circular plates. The direct and simple finite element formulations developed by Rao and Raju (1983) were used. Their result showed that in post buckling behavior of moderately thick circular plates subjected to either thermal loads (uniform temperature rise) or mechanical loads (uniform compressive load at the boundary) with more practically realistic elastic edge restraint, post buckling behavior of a circular plate with thermal loading was entirely different from that of a plate with mechanical loading when spring stiffeners increased.

Attalla (2002) studied the inelastic buckling strength of unsymmetrical tapered plates and subjected the plates with five different in-plane compressive loads to see if a simple design guideline would be obtained. The boundary conditions employed included simple support and fixed support. It was assumed in this work that all the plates had initial imperfection. The result showed that buckling strength of a tapered plate with tapering ratio of 0.5 was found to be more than 50% less than that of a rectangular plate in most cases. It also showed that reversing the direction of the in-plane linearly varying compressive stress dramatically affected the buckling strength of the tapered plate.

In another work, Audoly, Roman and Pocheau (2002) studied the nonlinear deformations of a long rectangular elastic plate clamped along its edges and submitted to in-plane biaxial compression and used the FÃ9ppl-von KÃirMÃin equation to predict various secondary buckling modes according to the applied longitudinal and transverse compressions. The result of a model experiment they carried out in a thin polycarbonate film showed that buckling patterns were in good agreement with theory. Zhu and Wilkinson (2007) tried to examine whether an equivalent circular Hollow section (CHS) can be used to model the local buckling of Elliptical Hollow section (EHS) when considering imperfections and nonlinear material properties. In doing this, they used finite element program ABAQUS. They examined the local buckling behavior of EHS with a range of aspect ratios from 1:1 CHS to 10:1. The analysis went through three stages. The first stage was elastic buckling with no material imperfection. The second stage considered inelastic material properties, and the third stage investigated how geometric imperfection affected the buckling modes. Their result showed that the use of an equivalent CHS was a reasonably good predictor of capacity of slender sections and the deformation capacity of compact sections.

The new version of differential quadrature method (D Q M) was used by Wang, Wang and Shi (2006) to obtain buckling loads of thin rectangular plates under non- uniform distributed in-plane loading. In doing this, they used two steps (i) solved a problem in plane stress elasticity to obtain the in-plane stress distribution, and(ii) solved the buckling problem under the loads obtained in step(i). Their results indicated that DQ method can be used to obtain buckling loads of plates with combinations of boundary conditions subjected to non-uniform distributed loading.

Eccher, Rasmussen and Zandonini (2007) described a method for testing the performance of finite elements with regard to shear locking. This method was called “monomial test”. Briassoulis (1988) introduced it and applied it to series of beam and plate elements. For the first time, this method was applied by Eccher and others to the isoparametric spline finite strip as the ameliorating influence of selective reduced integration.

In their work Yu and Schafer (2007) analyzed the effect of longitudinal stress gradients on the elastic buckling of thin isolated plate. They considered two types of thin plates: (i) a plate simply supported on all four edges and rotationally restrained on two longitudinal edges; and (ii) a plate simply supported on three edges with one longitudinal edge free and the opposite longitudinal edge rotationally restrained. During the analyses, they derived a semi analytical method and used it to calculate the elastic buckling stress of both types and rectangular thin plates subjected to non uniform applied longitudinal stresses. In order to validate the model, they used a finite element analysis (using ABAQUS). The result helped them to establish a better understanding of the effect of longitudinal stress gradients on the elastic buckling of thin plates.

Eccher, Rasmussen and Zandonini (2007) in another report studied the application of the isoparametric spline finite strip in geometric non linear analysis of perforated folded plate structures. They employed kinematics, strain-displacement and constitutive assumptions and applied them to the spline finite strip method. Furthermore, they derived the tangential and secant stiffness matrices by applying the equilibrium condition and its incremental form. They analyzed classical non-linear complex plate and shell problems and compared the solutions with exact solutions or with well established numerical results in order to demonstrate the reliability of the method.

2.3 BOUNDARY CONDITIONS

There are two types of boundary conditions namely, the essential and the non-essential boundary conditions (Bleich, 1952). The essential boundary conditions are those conditions that relate to the shape or geometry of the continuum at the boundaries. Generally, they are termed geometric or kinematic boundary conditions. Examples of them are deflection and rotation at the boundaries of the continuum. On the other hand, non-essential boundary conditions are those conditions that relate to the mechanical behavior of the continuum. They are called free, natural, force or dynamic boundary conditions by various authorities. In most cases, they are the conditions of bending moment and shear force at the boundaries of the continuum.

2.4 ELASTIC BUCKLING ANALYSIS OF THIN PLATES

Buckling of the plates could be studied, in general perspective, using the equilibrium approach or the energy approach (Iyengar, 1988). However, three approaches were identified in structural mechanics. They are equilibrium, energy and numeric approaches (Reddy, 1984). Equilibrium approach was also regarded as Euler approach. It sums all the forces acting on a continuum to zero. This summation was referred to the governing equation. It could either be ordinary differential equation or partial differential equation (Ugural, 1999).

Energy approach, on the other hand, sums all the work (strain energy and potential energy or external work) on the continuum to be equal to total potential energy (Iyengar, 1988). The numeric approach, depending on a particular method, will model the governing equation or the energy equation to approximate the solution of plate.

2.4.1 EQUILIBRIUM APPROACH

The equilibrium approach, commonly known as Euler approach, uses the equilibrium of forces to formulate the governing equation. This equation can be ordinary differential equation (for line continuum) or partial differential equation for plate. Governing partial differential equation for plate was given by Kirchoff (1877) and Venant (1883) as:

$$P + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = 0 \quad (2.4)$$

For plate that is only under in-plate loads, P will be dropped from eq(2.4) and the general equation becomes

$$N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = 0 \quad (2.5)$$

This equation could be solved by direct integration. The integration of this type of differential equation would yield a number of constant terms. Determination of the true values of these constant terms would lead to the completion of the solution of the differential equation (Szilard, 1974; Vinson, 1974; Mansfield, 1964 and Donnell 1976). The boundary conditions of the continuum would help in the determination of the constants. This approach of analysis of thin plates in elastic stability is not easy. For simple cases (like a rectangular plate simply supported all round, and loaded with in-plane load only along one axis) it could be employed to find exact solution of thin plates in elastic stability. However, for most practical cases in real time situations, this approach could be very complex and sometimes impossible (Reddy, 2002).

2.4.2 ENERGY APPROACH

When the solution from using the governing differential equation is becoming intractable, approximate means would be adopted. Energy approach has the inherent characteristic to be used as an approximate method. As mentioned above, this approach uses the total potential energy of a system to determine the unknown force or displacement at a specified point along or on the continuum. Typical examples of energy method include virtual work, least work, Castigliano, Betti, Maxwell etc (Richards, 1977; Shames and Dym, 1985; Davies, 1982). These energy methods yield approximate solution. The accuracy of theses methods depends on the closeness between the approximate deflection function and exact deflection function. Hence, there is the need for variational principle. This principle makes use of total potential functional. In variational calculus methods, the functional was first reduced to the governing differential equation and its boundary condition. Then the solution of the problem was sought. This was an indirect way of solving a problem (Arthurs, 1975). What this meant was that, one begins with the total potential energy functional and goes down to governing differential equation and its boundary condition, and finally solves the problem. It was obvious here that one had gone back to initial equilibrium approach from energy approach and thus met the same old problem. This was the problem that led to the development direct variational principle.

2.4.3 DIRECT VARIATIONAL PRINCIPLE

Moving back to governing equation from total potential energy function in variational principle gave rise to alternate solution of practical problems. Substituting an assumed shape function into the functional and working directly to minimize the functional without getting back to the governing equation is what was known today as direct variational principle (Elsgolts, 1980 and Arthurs, 1975). Examples of direct methods of variational calculus in elastic stability of plates included Rayleigh-Ritz, Euler finite difference and Kantorovich's methods (Rao, 1989; Cook, Malkus and Plesha, 1989; Elsgolts, 1980 and Arthurs, 1975).

Starting point of direct variational principle was the total potential energy functional. It was an integral expression that contains, implicitly, the governing differential equation. This was a weak form of expressing a problem. It was a weak form because it stated the conditions that must be met over a domain (integrating through out the span of the continuum). It did not state the conditions that must be met at every point on the continuum. The strong way of expressing a problem in structural mechanics was the use of governing differential equation and the corresponding boundary conditions (Langhaar, 1962).

That a functional implicitly contained the governing differential equation meant that the end result of using a functional in a variational calculus was the governing differential equation and the non essential (natural) boundary conditions (Lanczos, 1949; Courant and Hilbert, 1953 and Washizu, 1968)

2.4.4 RAYLEIGH-RITZ METHOD

In this method, a trial shape function is assumed. However, care must be taking in selecting the trial function. This shape function should as much as possible be close to the deformed shape function of the continuum. The assumed shape function can be formulated from Fourier series or Taylor-Maclaurin series. If the chosen shape function is far from being a good approximation of the exact deformed shape function then the solution will be far from being an approximation of the exact solution. If, by accident, the exact shape function is assumed then the solution will correspond to exact solution. This shape function, $w(x) = \sum_{i-1}^{n} ai\, wi\, (x)$, is a function of generalized coordinate, ai and coordinate, wi.

Having done with the selection of shape function, substitute the boundary conditions into it to reduce it to a peculiar shape function. Substitute this particular shape function into the total potential energy functional. This functional will be integrated over the domain to reduce it to a function (F (ai)) of generalized coordinates. The resulting function will be partial differentiated with respect to the generalized coordinates and equated to zero.

$$\frac{\partial F(ai)}{\partial a_1} = \frac{\partial F(ai)}{\partial a_2} = \cdots = \frac{\partial F(ai)}{\partial a_n} = 0 \qquad (2.6)$$

This will give n linear equations that will be solved simultaneously to get characteristic equation. The roots of the equation are the Eigen-values. The smallest Eigen-value is the critical (buckling) load of the plate. (Rayleigh, 1945; Ritz, 1909; Elsgolts, 1980; Arthurs, 1975)

This method has a good advantage over the equilibrium approach in the sense that it is able to give solutions (though approximate) to some problems that are intractable by equilibrium approach. The functional used is of lower order than the governing differential equation. However, the disadvantage is that it only gives approximate solutions. All the same, it has the ability to converge to exact solution as the number of terms in the assumed shape function tends to infinity. The fewer the number of terms in the shape function the easier the computation and the worse the solution. On the other hand, the more the number of terms in the shape function the harder the computation and the better the solution.(Leipholz, 1977; and Mikhlin, 1964)

2.5 TAYLOR-MCLAURIN SERIES

Any representation of a function, F(x), as an infinite sum of terms calculated from the values of the derivatives of the function at single point, x_0 is known as Taylor series. This is expressed mathematically as:

$$\sum_{n=0}^{\infty} \frac{F^{(n)}(x_0)}{n!}(x - x_0)^n$$

$F^{(n)}(x_0)$ meant the nth derivative of F(x) determined at point x_0; n! meant factorial of n. Expanding the expression will give:

$$\frac{F^0(x_0)}{0!}(x-x_0)^0+\frac{F^I(x_0)}{1!}(x-x_0)^1+\frac{F^{II}(x_0)}{2!}(x-x_0)^2+\frac{F^{III}(x_0)}{3!}(x-x_0)^3+\cdots$$

Where $F^0(x_0)=F(x_0)$. That is zero differentiation of $F(x_0)$ or $F(x_0)$ without differentiation. $F^I(x_0)$, $F^{II}(x_0)$ and $F^{III}(x_0)$ are respectively, first, second and third derivatives of $F(x_0)$.

$$0!=1!=1;\ (x-x_0)^0=1;(x-x_0)^1=(x-x_0)$$

Thus the Taylor series can be written as:

$$F(x_0)+F^I(x_0)(x-x_0)+\frac{F^{II}(x_0)}{2!}(x-x_0)^2+\frac{F^{III}(x_0)}{3!}(x-x_0)^3+\cdots$$

(Stroud, 1982; Tranter and Lambe, 1973; Boyce and Diprima, 1977)

If the point of consideration, x_0 is equal to zero then the series becomes Maclaurin series and is expressed as:

$$F(0)+F^I(0)x+\frac{F^{II}(0)}{2!}x^2+\frac{F^{III}(0)}{3!}x^3+\cdots$$

It can be said that Maclaurin series is a special Taylor series when ever $x_0 = 0$. This can be called Taylor-Maclaurin series. It is seen here that Taylor-Maclaurin series is nothing but power series of x representing the function, f(x). Expressing a function, f(x) as an infinite series in powers of x - x_0, where $x_0 = 0$ is called power series. The other way round, power series is defined as Taylor-Maclaurin series of a real or complex function, f(x) that is infinitely differentiable in a neighborhood of a real or complex number, x_0. Taking

$\frac{F^n(x_0)}{n!}$ as a_n and x_0 as zero then the Taylor Maclaurin series can be rewritten as:

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \qquad (2.7)$$

(Thomas and Finney, 1996; Greenberg, 1998).

It is a common knowledge in structural engineering and engineering mechanics that the deformed shapes of continua under load take the form of sinusoidal shapes of sine or cosine curves with angles measured in radians. Some times the shape can be exactly sine curve and some other times just been closed to it. Hence, trigonometric functions are normally being used in shape function approximation. Taylor-Maclaurin series have the property to converging into many functions, including trigonometric functions. This convergence can be rapid (in the case of trigonometric function approximation) if $-1 < x < 1$. If the magnitude of x is greater than one then more terms will be used to ensure convergence (Edward, 2004).

2.6 BUCKLING MODE OF A RECTANGULAR PLATE

A square or near square thin plate always buckles in the first mode of half wave length. If the plate is loaded in the two directions (x and y) then the first mode of failure shape will be half wave length in x and half wave length in y. That is to say the wave length is equal 2a or 2b, where "a" and "b" are the total lengths of the plate in x and y directions respectively. The failure modes M (for x direction) and N (for y direction) are equal to one each. If the plate buckles in the second mode then M = N = 2. But if the load is only in x direction and the plate buckles in the nth mode then N= 1 and M = n. This implies that when a rectangular plate is loaded only in one direction, the buckling mode in the unloaded direction is always equal to one. If the loaded direction (in the case of loading only in one direction) is always taking as x direction, then N is equal to one, N = 1 (Murray, 1984). Aspect ratio, P is equal to $\frac{a}{b}$. But Iyengar (1988) gave the relationship between aspect ratio and buckling mode, M (when N = 1) as P = $\sqrt{M\,(m+1)}$. This relationship is the value of aspect ratio when the buckling is changing from one mode to the immediate higher or lower mode. That is to say it is the transitory aspect ratio (the aspect ratio between one mode and the immediate next). For instance P = $\sqrt{2}$ is the aspect ratio between modes one and two. However, from the works of Murray (1984) and Timoshenko and Gere (1961) it was gathered that when none of the four edges of a rectangular thin plate is free of support, aspect ratio, P can be defined in terms of a, b, M and N as P = $\frac{a}{b} \equiv \frac{aN}{bM}$. This (according to Timoshenko and Gere, 1961) was to comply with there findings that a long rectangular thin plate (having all the four edges simply supported) that is loaded axially in the longitudinal direction, behave in a similar way to a square plate except that it buckles into almost several square panels.

2.7 EIGEN-VALUE PROBLEM

One of envisaged problem in this research is the handling of the resulting stability matrix equations. This is because the resulting Eigen-value problem will require consistent mass matrix as against lumped mass matrix. In structural mechanics, shape functions are usually assumed to approximate the deformed shape of the continuum. If the assumed shape function is the exact one, then the solution will converge to exact solution. The stiffness matrix [K] is formulated using the assumed shape function. In the same way, the mass matrix, [M] and the geometric stiffness matrix, [Kg] will be formulated. The mass matrix and the geometric stiffness matrix formulated in this way are called consistent mass matrix and consistent geometric stiffness matrix respectively (Paz, 1980 and Geradin, 1980). Owen (1980) gave the equation to be:

$[K]\{X\} = \omega^2[M]\{X\}$, which is equal to

$$([K] - \omega^2[M])\{X\} = 0$$

It is easy to determine the Eigen-values of this equation when the size of the square matrix is not more than 3 x 3. Geradin (1980) noted that significant amount of computational effort is required for the Eigen-value problems using consistent mass matrix. Because of this difficulty, many analysts preferred using lump mass matrix to consistent mass matrix. The works of Key and Krieg (1972) and Key (1980) showed that the difference between the solutions from lump mass and consistent mass is very significant. Since the difference in the solutions is high, it will be foolhardy to rely upon the solutions from lump mass just because it is easy to solve. Sheik et al (2004) recommended efficient mass lumping scheme to form a mass matrix having zero mass for the internal nodes. This, according to them, would help facilitate condensation of the structural matrix. The use of lump mass matrix will transform the consistent matrix equation to:

$$([K] - \omega^2 m[I])\{X\} = 0$$

where [I] is the identity matrix. This equation of lump mass is simply written as

$$(A - \lambda I)X = 0$$

Where A is a square matrix, λ is a scalar number called Eigen-value or characteristic value of matrix A, I is identity matrix and X is the eigenvector (Stroud, 1982 and James, Smith and Wolford, 1977). Efforts were made to search for literature where the solution of Eigen-value matrices have consistent mass. Many methods of solution exist for equations of lump mass. Some of the methods include Jacobi method, polynomial method, iterative methods and Householder's method (Greenstadt, 1960; Ortega, 1967 and James, Smith and Wolford, 1977). Iterative method means method based on matrix – vector multiplication. Some of the iterative methods include power method, inverse iteration method (Wilkinson, 1965), Lanczos method (Lanczos, 1950), Arnoldi method (Arnoldi, 1951, Demmel, 1997, Bai et al, 2000, Chatelin, 1993, and Trefethen, and Bau, 1997), Davidson method, Jacobi-Davidson method (Hochstenbach and Notay, 2004 and Sleijpen and van der Vorst, 1996), Minimum Residual method, generalized minimum residual method (Barrett et al, 1994). Others are Multilevel Preconditioned Iterative Eigensolvers (Arbenz and Geus, 2005), Block Inverse-Free Preconditioned Krylov Subspace method (Quillen and Ye, 2010), Inner-outer Iterative method (Freitag, 2007), Adaptive Inverse Iteration method (Chen, Xu and Zou, 2010). Unfortunately, these methods, except the polynomial method, can only be used for equations of lump mass. They can not handle equation of consistent mass. Polynomial method becomes very difficult to use when the size of the matrix exceeds 3 x 3. This predicament made the proposed study to seek to devise a way of solving Eigen-value equations of consistent mass matrices when the sizes of the matrices exceed 3 x 3.

2. 8 SOLUTIONS OF RECTANGULAR THIN PLATES OF VARIOUS BOUNDARY CONDITIONS.

Some scholars had in the past found solutions (some exact and some approximate) to rectangular thin plates that were buckling under in-plane forces. Their solutions were presented according to different boundary conditions.

2. 8. 1 SOLUTION OF RECTANGULAR PLATE WITH THE FOUR EDGES SIMPLY SUPPORTED, SSSS

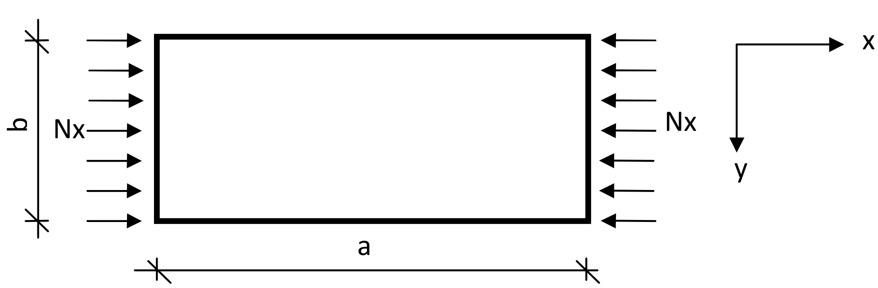

Figure 2.1: SSSS plate with in-plane load

Iyenga (1988) used the governing partial differential equation to get the exact solution to be

$$(N_x)_{cr} = \frac{K D \pi^2}{b^2}$$

Where K = $(M/P + N^2 P/M)^2$

If N = 1 then K = $(M^2/P^2 + P^2/M^2 + 2)$

Timoshenko and Krieger (1970) used the variation of total potential energy method, and assume a shape function of

$$W = \sum_{m=1}^{\infty} . \sum_{n=1}^{\infty} a_{MN} . Sin(\frac{m\pi x}{a}) . Sin\left(\frac{n\pi y}{b}\right) \text{ to Solve the same problem.}$$

His result showed that

$$K = (\frac{Mb}{a} + \frac{a}{Mb})2 = \frac{M^2}{P^2} + \frac{P^2}{M^2} + 2$$

2. 8. 2 SOLUTION OF RECTANGULAR PLATE WITH THE FOUR EDGES CLAMPED, CCCC

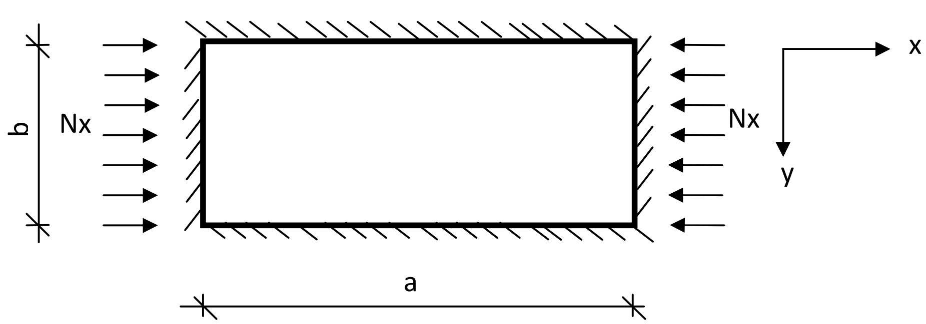

Figure 2.1: CCCC plate with in-plane load

Iyengar (1988) used the variation of total potential energy method, and assumed a shape function of W= A (1- Cos $2\pi R$) (1- Cos $2\pi Q$), where

$R = {}^{x}/_{a}\ and\ Q = {}^{y}/_{b}$, to solve the problem. He got, for first buckling mode

$$(Nx)cr = \frac{K\,\pi^2\,D}{b^2}, \text{Where K} = 4/P2 + 4P2 + 2.667.$$

Fok (1980) used finite difference method and found the solution for a square plate of first buckling mode as K = 10.12.

Levy (1942) used an infinite series for the shape function and got K for a square thin rectangular plate as K = 10.07.

Timosshenko (1936), Heck and Ebner (1936) and Maulbetsh (1937) in their separate studies found K to be related to aspect ratio, P as follows:

P	1	2	3	∞
K	9.362	8.146	7.781	6.967

2. 8. 3 SOLUTION OF RECTANGULAR PLATE WITH TWO OPPOSITE EDGES CLAMPED AND THE OTHER TWO SIMPLE SUPPORTED

FOR CSCS

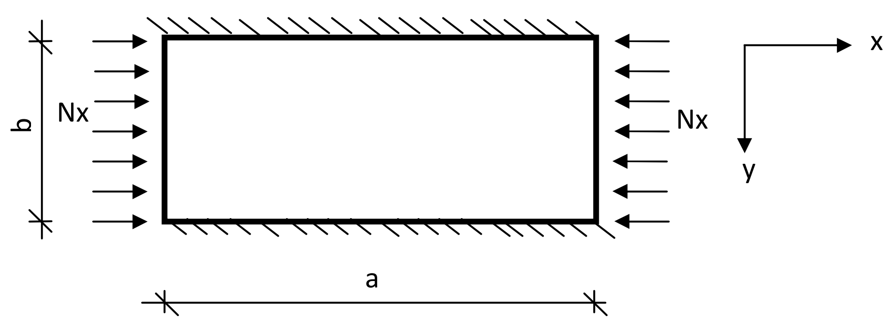

Figure 2.3: CSCS plate with in-plane load

Iyenger (1988) found K from using energy approach and shape function of

W = A Sin $m\pi R\,(1 - Cos\ 2\pi Q)$ to be

$$K = \frac{M^2}{P^2} + 5.3333\,\frac{P^2}{M^2} + 2.6667$$

Fok (1980) used finite difference method to find K to be 6.98 for aspect

ration of 3.

Timoshenko (1936) and Heck and Ebner (1935) in separate studies found K for various aspect ratios, P to be:

P	0.4	0.5	0.6	0.7	0.8	1	1.2	1.4	1.6	1.8	2.1	∞
K	9.44	7.68	7.05	7.00	7.30	7.68	7.05	7.00	7.295	7.05	7.00	6.97

FOR SCSC

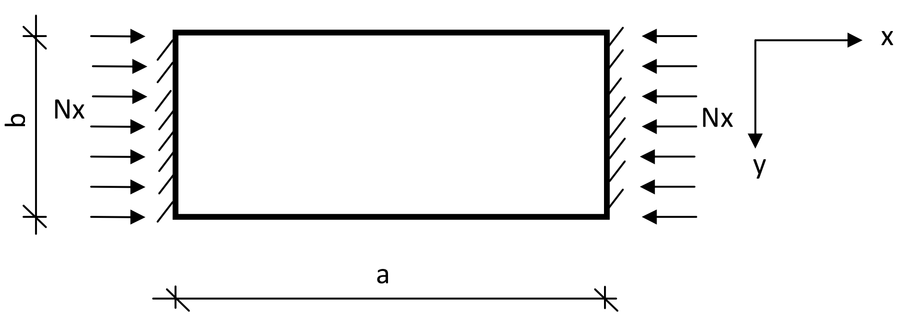

Figure 2.4: SCSC plate with in-plane load

Though, Iyenger (1988) did not continue his solution to take care of a case like that shown in the above figure, substituting his variables, w, w^{IR}, w^{IQ}, w^{IIR}, w^{IIQ} and w^{IIRQ} into the energy equation he used gave:

$$K = \frac{4M^2}{P^2} + \frac{0.75P^2}{M^2} + 2$$

Where w^{IR} means differentiating w partially once with respect to R. w^{IQ} means differentiating w partially once with respect to Q. w^{IIR} means differentiating w partially two times with respect to R. w^{IIQ} means differentiating w partially two times with respect to R. w^{IIRQ} means differentiating w partially once with respect to R and partially once with respect to Q.

Timoshenko (1936) gave his own solution as

P	0.6	0.8	1	1.2	1.4	1.6	1.7	1.8	2	2.5	∞
K	13.37	8.73	6.74	5.84	5.45	5.34	5.34	5.18	4.85	4.52	4.41

Michelutti (1976) used Runge-Kutta method to get:

P	0.66
K	6.97

Fok (1980) used finite difference method to get:

P	1	3
K	6.85	4.28

2. 8. 4 SOLUTION OF RECTANGULAR PLATE WITH ONE EDGE CLAMPED AND THE OTHER EDGES SIMPLE SUPPORTED, CSSS

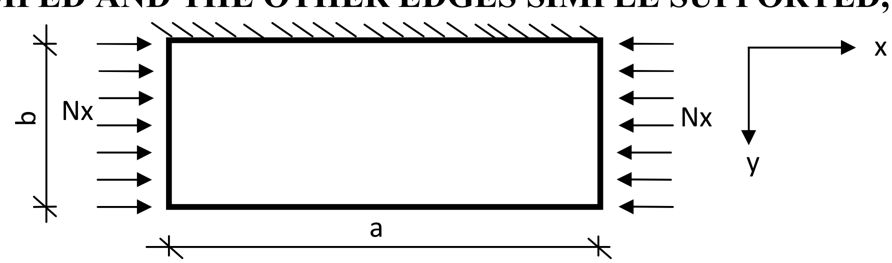

Figure 2.5: CSSS plate with in-plane load

Fok (1980) used finite difference method to get:

P	1
K	5.4

Michelutti (1976) used Runge- Kutta method to get:

P	0.79
K	5.41

2. 8. 5 SOLUTION OF RECTANGULAR PLATE WITH THREE EDGES CLAMPED AND ONE EDGE SIMPLE SUPPORTED, CCSC

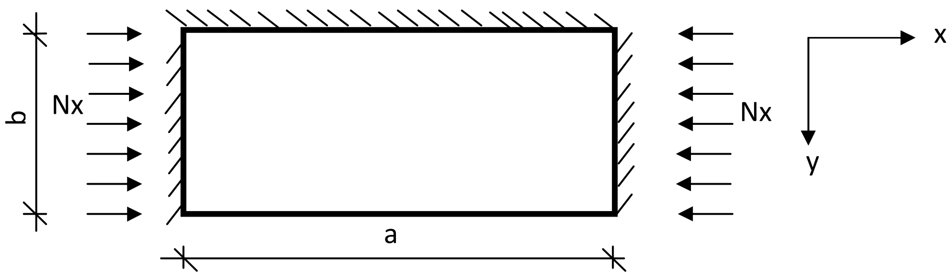

Figure 2.6: CCSC plate with in-plane load

Fok (1980) used finite difference method to get:

P	3
K	5.6

2. 8. 6 SOLUTION OF RECTANGULAR PLATE WITH THREE EDGES SIMPLY SUPPORTED AND ONE EDGE FREE, SSFS

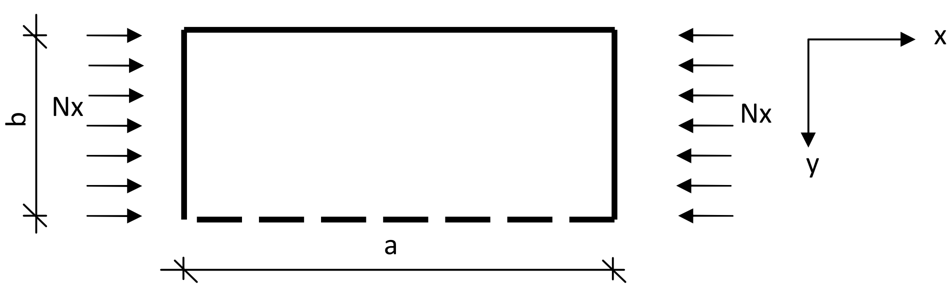

Figure 2.7: SSFS plate with in-plane load

Fok (1980) used finite difference method to get:

P	1	2	3
K	1.4	0.675	0.53

Michelutti (1976) used Runge- Kutta method to get:

P	5
K	0.46

Timoshenko (1936) got:

P	0.5	1	1.2	1.4	1.6	1.8	2	2.5	3	4	5
K	4.401	1.44	1.14	0.953	0.835	0.756	0.698	0.61	0.564	0.517	0.506

2. 8. 7 SOLUTION OF RECTANGULAR PLATE WITH ONE EDGE CLAMED, TWO OPPOSITE EDGES SIMPLY SUPPORTED AND ONE EDGE FREE, CSFS

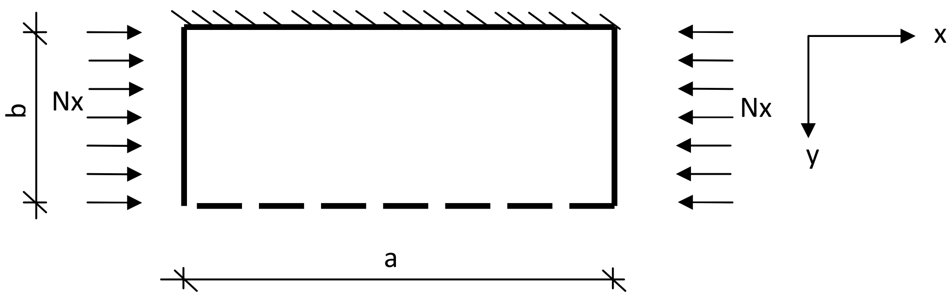

Figure 2.8: CSFS plate with in-plane load

Fok (1980) used finite difference method to get:

P	1	3
K	1.64	1.25

Michelutti (1976) used Runge-Kutta method to get:

P	1.63
K	1.25

Timoshenko (1936) got

P	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2	2.2	2.4
K	1.7	1.56	1.47	1.41	1.36	1.34	1.33	1.33	1.34	1.36	1.39	1.45	1.47

Iyengar (1988) got:

P	0.2	0.4	0.8	1	1.414	2	2.45	3.2	3.464	4.5
K	23.35	6.34	2.16	1.75	1.43	1.47	1.64	1.39	1.41	1.4

2. 8. 8 SOLUTION OF RECTANGULAR PLATE WITH TWO OPPOSITE EDGES CLAMED, ONE EDGE SIMPLY SUPPORTED AND ONE EDGE FREE, SCFC

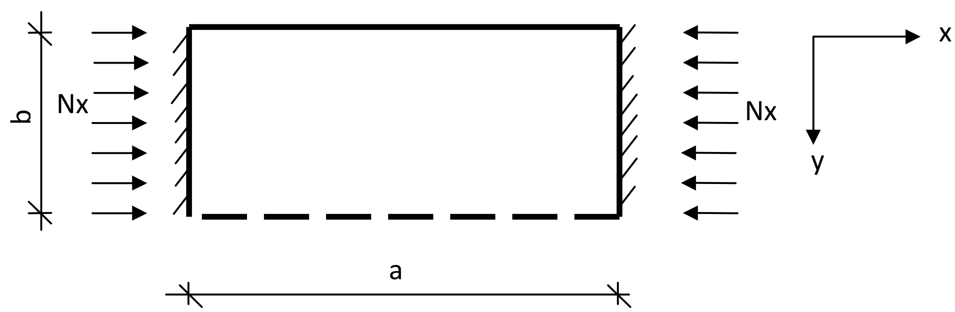

Figure 2.9: SCFC plate with in-plane load

Fok (1980) used finite difference method to get:

P	1	2	3
K	2.52	1.954	0.85

2.9 NATURE OF SOLUTION FROM DIRECT VARIATION METHODS

Let w be the exact shape function. Assume this function is an infinite series. The function can be approximated by truncating the infinite series to get a finite one. Let this finite series approximation of the exact shape function be W. Total potential energy functional is a function of shape function. Hence, it can be a function of w, J(w) or function of W, J (W_n).

The solution from minimizing the functional, J is such that J (w) $\leq$ J (W_n). The implication of this condition is that

$$J\ (w) < \lim_{n \to \infty} J(W_n)$$

Where n is a finite number. In other words, the solution using finite approximation of the finite series (exact shape function) is always greater than the exact solution. Many authorities agreed to this. Generally, every solution from direct variation principle is an upper bound solution (that is higher in value than exact value). However if the exact shape function is used, the solution will correspond to exact solution. Hence, when a solution from direct variation principle is less than exact solution, something must be wrong in the procedure (Arthurs, 1975; Elsgolts, 1980; Iyengar, 1988).

CHAPTER THREE
METHOD

3.1 TOTAL POTENTIAL ENERGY FUNCTIONAL

Displacement in the vertical (z) direction, as shown in figure 3.1, is w. "w" varies with respect to x and y. Displacement in x direction is u while that in y direction is v. Initial displacements in x, y and z directions are u_o, v_o and w_o. In this problem the value of w depends on the values of x, y and z. Hence, w and w_o were said to be functions of x and y. "w" can be written as w(x,y) and w_o can be written as $w_o(x,y)$. It is a common knowledge that:

$$stress = strain\ x\ modulus\ of\ elasticity\ .$$

That is

$$\sigma = \in .E \qquad (1)$$

Hence,

$$\sigma_x = \in_x .E\ ;\ \ \sigma_y = \in_y .E\ \ \ and\ \ \sigma_z = \in_z .E \qquad (2)$$

$$\sigma_z = weight\ of\ material * material\ thickness = eg * z \qquad (3)$$

Where "*e*" is the material density, g is the acceleration due to gravity and z varies from 0 to h.

$$\sigma_z = e.g.z = \in_z .E \qquad (4)$$

Therefore,

$$\epsilon_z = \frac{e.g.z}{E} \tag{5}$$

Strain is the ratio of change in size (shape) to original size (shape). Hence

$$\epsilon_x = \frac{du}{dx}, \quad \epsilon_y = \frac{dv}{dy} \quad and \quad \epsilon_z = \frac{dw}{dz} \tag{6}$$

That is to say

$$\epsilon_z = \frac{dw}{dz} = \frac{e.g.z}{E} \tag{7}$$

Hence,

$$dw = \frac{e.g.z}{E}\, dz \tag{8}$$

$$w(x,y) = \int dw = \int \frac{e.g.z}{E} dz = \frac{e.g.z^2}{2E} + w_o(x,y)$$

$$w(x,y) = \frac{e.g.z^2}{2E} + w_o(x,y) \tag{9}$$

In this problem, shear strains in x - z and y - z were equal to zero. That is,

$$\gamma_{xz} - 0 - \frac{du}{dz} + \frac{dw}{dx} \tag{10}$$

$$\gamma_{yz} = 0 = \frac{dv}{dz} + \frac{dw}{dy} \tag{11}$$

This means that

$$\frac{\partial u}{\partial Z} = -\frac{\partial w(x,y)}{\partial x} \tag{12}$$

and

$$\frac{\partial v}{\partial Z} = -\frac{\partial w(x,y)}{\partial y} \tag{13}$$

$$\partial u = -\frac{\partial w(x,y)}{\partial x} * \partial Z = \frac{-\partial}{\partial x}\left(\frac{e.g.z^2}{2E} + w_0(x,y)\right) dz$$
$$= -\frac{\partial w_0(x,y)}{\partial x}\,\partial z \tag{14}$$

$$u = \int \partial u = \int -\frac{\partial w_0(x,y)}{\partial x}\,\partial z = -Z\,\frac{\partial w_0(x,y)}{\partial x} + u_0 \tag{15}$$

Similarly,

$$v = -Z\,\frac{\partial w_0(x,y)}{\partial y} + v_0 \tag{16}$$

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(-Z\,\frac{\partial w_0(x,y)}{\partial x} + u_0\right) = -Z\,\frac{\partial^2 w_0(x,y)}{\partial x^2} \tag{17}$$

Similarly

$$\epsilon_y = -Z\,\frac{\partial^2 w_0(x,y)}{\partial y^2} \tag{18}$$

Shear strain in x – y is given as

$$\gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx} \tag{19}$$

$$= \frac{\partial}{\partial y}\left(-Z\,\frac{\partial w_0(x,y)}{\partial x} + u_0 \right) + \frac{\partial}{\partial x}\left(-Z\,\frac{\partial w_0(x,y)}{\partial y} + v_0 \right)$$

That is

$$\gamma_{xy} = -2Z\,\frac{\partial^2 w_0(x,y)}{\partial x \partial y} \qquad (20)$$

It can be seen that all strains, ε_x, ε_y and γ_{xy} are function of $w_0(x,y)$. That is to say they depend on $w_0(x,y)$. The relationship between vertical displacement, w, longitudinal dimension, x, and elastic deformation curve, s is shown in figure 3.1.

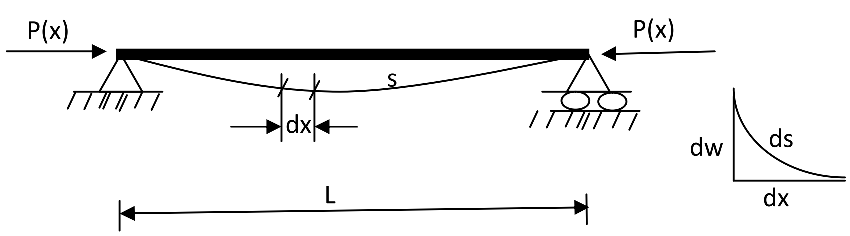

Figure 3.1a: Deformed continuum

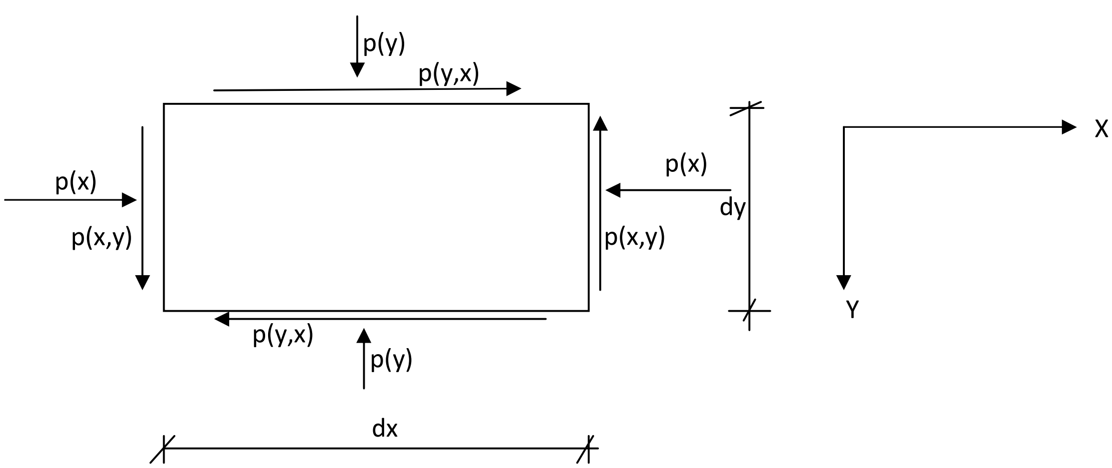

Figure 3.1b: Plan of continuum

Pythagoras theorem was used to relate s, w and x as: $ds^2 = dw^2 + dx^2$

$$\left(\frac{ds}{dx}\right)^2 = \left(\frac{dw}{dx}\right)^2 + 1$$

$$\frac{ds}{dx} = \sqrt{\left(\frac{dw}{dx}\right)^2 + 1}$$

Binomial theorem was used here to expand the above expression as

$$\sqrt{\left(\frac{dw}{dx}\right)^2 + 1} = 1 + \frac{1}{2}.\left(\frac{dw}{dx}\right)^2 + \frac{\frac{1}{2}.-\frac{1}{2}}{2!}\left(\frac{dw}{dx}\right)^4 + \frac{\frac{1}{2}.-\frac{1}{2}.-\frac{3}{2}}{3!}\left(\frac{dw}{dx}\right)^6 + \dots$$

$$= 1 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - \frac{1}{8}\left(\frac{dw}{dx}\right)^4 + \frac{1}{16}\left(\frac{dw}{dx}\right)^6 + \dots$$

$\left(\frac{dw}{dx}\right)^2$ was a small quantity, its higher powers was negligible . Hence,

$$\sqrt{\left(\frac{dw}{dx}\right)^2 + 1} = 1 + \frac{1}{2}.\left(\frac{dw}{dx}\right)^2 .\ Thus, \qquad \frac{ds}{dx} = 1 + \frac{1}{2}.\left(\frac{dw}{dx}\right)^2$$

$$ds = \left(1 + \frac{1}{2}.\left(\frac{dw}{dx}\right)^2\right)dx$$

$$S = \int_0^L ds = \int_0^L \left(1 + \frac{1}{2}.\left(\frac{dw}{dx}\right)^2\right)dx = L + \int \frac{1}{2}.\left(\frac{dw}{dx}\right)^2 dx\ .Thus$$

$$= L + \int 1 + \frac{1}{2}.\left(\frac{dw}{dx}\right)^2 dx\ .Thus,\ S - L = \frac{1}{2}\int \left(\frac{dw}{dx}\right)^2 dx \qquad (21)$$

Change in length of the continuum, Δ is equal to

$$\varDelta = S - L = \frac{1}{2}\int \left(\frac{dw}{dx}\right)^2 dx \qquad (22)$$

External work, EW is the product of applied force and change in length.

$$EW_{(x)} = P_{(x)}.\Delta$$

$$= P_{(x)}.\frac{1}{2}\int \left(\frac{dw}{dx}\right)^2 dx$$

This is

$$EW_{(x)} = \frac{1}{2}\int P_{(x)} \left(\frac{dw}{dx}\right)^2 dx \qquad (23)$$

$$EW_{(y)} = \frac{1}{2}\int P_{(y)} \left(\frac{dw}{dy}\right)^2 dy \qquad (24)$$

$$EW_{(x,y)} = \frac{1}{2}\int P_{(x,y)} \frac{dw}{dx}.\frac{dw}{dy} dx \qquad (25)$$

$$EW_{(y,x)} = \frac{1}{2}\int P_{(y,x)} \frac{dw}{dy}.\frac{dw}{dx} dy \qquad (26)$$

But

$$P_{(x)} = p_{(x)} dy$$

$$P_{(y)} = p_{(y)} dx$$

$$P_{(x,y)} = pdy$$

$$P_{(y,x)} = p_{(y,x)} dx$$

Thus,

$$EW_{(x)} = \frac{1}{2}\int p_{(x)} \left(\frac{dw}{dx}\right)^2 dx\,.\,dy$$
$$= \frac{1}{2}\int p_{(x)} \left(\frac{dw}{dx}\right)^2 dA \qquad (27)$$

$$EW_{(y)} = \frac{1}{2}\int p \left(\frac{dw}{dy}\right)^2 dy\,.\,dx$$
$$= \frac{1}{2}\int p_{(y)} \left(\frac{dw}{dx}\right)^2 dA \qquad (28)$$

$$EW_{(xy)} = \frac{1}{2}\int p_{(xy)} \frac{dw}{dx}\cdot\frac{dw}{dy}\,.\,dx\,.\,dy$$
$$= \frac{1}{2}\int p_{(xy)} \frac{dw}{dx}\cdot\frac{dw}{dy} dA \qquad (29)$$

$$EW_{(yx)} = \frac{1}{2}\int p_{(yx)} \left(\frac{dw}{dx}\right)^2 dx\,.\,dy$$
$$= \frac{1}{2}\int p_{(yx)} \left(\frac{dw}{dx}\right)^2 dA \qquad (27)$$

$$EW_{(x)} = \frac{1}{2}\int p_{(x)} \frac{dw}{dy}\cdot\frac{dw}{dx}\,.\,dy\,.\,dx$$
$$= \frac{1}{2}\int p_{(x)} \frac{dw}{dy}\cdot\frac{dw}{dx} dA \qquad (30)$$

$$EW = EW_{(x)} + EW_{(y)} + EW_{(xy)} + EW_{(yx)} \qquad (31)$$

Substituting equations 27, 28, 29 and 30 into 31 gave

$$EW = \frac{1}{2}\int \left(p_{(x)} \left(\frac{dw}{dx}\right)^2 + p_{(y)} \left(\frac{dw}{dy}\right)^2 + p_{(xy)} \frac{dw}{dx}\cdot\frac{dw}{dy} + p_{(yx)} \frac{dw}{dy}\cdot\frac{dw}{dx}\right) dA$$

But, $P_{(y,x)} = P_{(y,x)}$. *Hence,*

$$EW = \frac{1}{2}\int_A \left(p_{(x)}\left(\frac{dw}{dx}\right)^2 + p_{(y)}\left(\frac{dw}{dy}\right)^2 + 2p_{(xy)}\frac{dw}{dx}.\frac{dw}{dy}\right)dA \qquad (32)$$

In direct variational principle, we make use of approximate deflection function, $w_0 \approx w$. Thus, equation 32 could be written as

$$EW \approx \frac{1}{2}\int_A \left(p_{(x)}\left(\frac{dw_0}{dx}\right)^2 + p_{(y)}\left(\frac{dw_0}{dy}\right)^2 + 2p_{(xy)}\frac{dw_0}{dx}.\frac{dw_0}{dy}\right)dA \qquad (32a)$$

The internal (strain) energy before loading was designated as U_1. After the loading, internal energy was designated as U_2. The average internal energy was designated as U. Thus,

$$U = \frac{U_1 + U_2}{2} \qquad (33)$$

Energy is the product of force and change in length.

That is, *Energy = Force x Δ* (34)

Force is the product of stress and area. That is,

$$Force = \sigma * A \qquad (35)$$

If an element of the plate was under consideration, then elemental force is equal to

$$Elemental\ force = \sigma * dA \qquad (36)$$

Change in length is the product of *linear strain* and *original length*. Hence, elemental change in length was given by,

$$d\Delta = \in . dL \qquad (37)$$

Substituting equations 36 and 37 into equation 34 gave the equation of elemental energy. Initial strain energy, u_1 was equal to zero. Hence, elemental final strain energy, dU_2 was given by

$$dU_2 = (\sigma . dA).(\in . dL) = \sigma . \in . dV \qquad (38)$$

$$Hence, \ dU = \frac{0 + \sigma . \in . dV}{2} = \frac{1}{2} \sigma . \in . dV \qquad (39)$$

$$Thus, \quad U - \frac{1}{2} \int \sigma . \in . dV \qquad (40)$$

$$U_x = \frac{1}{2} \int \sigma_x \in_x dV \qquad (41)$$

$$U_y = \frac{1}{2} \int \sigma_y \in_y dV \qquad (42)$$

$$U_{xy} = \frac{1}{2} \int \tau_{xy} \gamma_{xy} \, dV \qquad (43)$$

$$U = U_x + U_y + U_{xy} \qquad (44)$$

$$Therefore, U = \frac{1}{2} \int (\sigma_x \in_x + \sigma_y \in_y + \tau_{xy} \gamma_{xy}) dV \qquad (45)$$

$$Total\ Potential\ energy, \quad \prod = U - EW \qquad (46)$$

Shear stress is the product of shear strain and shear modulus. That is

$$\tau_{xy} = \gamma_{xy}.G \qquad (47)$$

According to Timoshenko and Goodier (1951),

$$G = \frac{E}{2(1+\mu)} \qquad (48)$$

Substituting equation 48 into equation 47 gave

$$\tau_{xy} = \gamma_{xy}.\frac{E}{2(1+\mu)} \qquad (49)$$

μ stands for Poisson ratio.

Applying direct stress, σ_x in a plate will cause the plate to strain in x, y and z directions as:

$$\epsilon_x = \frac{\sigma_x}{E} \qquad (50)$$

$$\epsilon_y = -\mu\,\frac{\sigma_x}{E} \qquad (51)$$

$$\epsilon_z = -\mu\,\frac{\sigma_x}{E} \qquad (52)$$

Applying direct stress, σ_y in a plate will cause it to strain in x, y and z directions as:

$$\epsilon_x = -\mu\,\frac{\sigma_y}{E} \qquad (53)$$

$$\in_y = \frac{\sigma_y}{E} \tag{54}$$

$$\in_z = -\mu \frac{\sigma_y}{E} \tag{55}$$

Applying direct stress, σ_z in a plate will cause it to strain in x, y and z directions as:

$$\in_x = -\mu \frac{\sigma_z}{E} \tag{56}$$

$$\in_y - -\mu \frac{\sigma_z}{E} \tag{57}$$

$$\in_z = \frac{\sigma_z}{E} \tag{58}$$

Adding equations. 50, 53 and 56 partially will give

$$\in_x = \frac{1}{E}\left(\sigma_x - \mu(\sigma_y + \sigma_z)\right) \tag{59}$$

Similarly

$$\in_y = \frac{1}{E}\left(\sigma_y - \mu(\sigma_x + \sigma_z)\right) \tag{60}$$

$$\in_z = \frac{1}{E}\left(\sigma_z - \mu(\sigma_x + \sigma_y)\right) \tag{61}$$

In a plate of in-plane stress, σ_z was taken as zero. Hence

$$\in_x = \frac{1}{E}\left[\sigma_x - \mu\sigma_y\right] \tag{62}$$

$$\in_y = \frac{1}{E}\left[\sigma_y - \mu\sigma_x\right] \tag{63}$$

Rearranging equations (62) and (63) respectively gave

$$\sigma_x = E \in_x + \mu\sigma_y \tag{64}$$

$$\sigma_x = \frac{\sigma_y - E \in_y}{\mu} \tag{65}$$

Equating equations 64 and 65 gave

$$E \in_x + \mu\sigma_y = \frac{\sigma_y - E \in_y}{\mu}$$

$$\mu E \in_x + \mu^2\sigma_y = \sigma_y - E \in_y$$

$$\sigma_y(1 - \mu^2) = E(\mu \in_x + \in_y)$$

$$\sigma_y = \frac{E}{1 - \mu^2}\left(\mu \in_x + \in_y\right) \tag{66}$$

Similarly

$$\sigma_x = \frac{E}{1 - \mu^2}\left(\mu \in_y + \in_x\right) \tag{67}$$

Substituting equations 17 and 18 into 67 gave

$$\sigma_x = \frac{E}{1 - \mu^2}\left(-Z\frac{\partial^2 w_0(x,y)}{\partial x^2} - Z\mu\frac{\partial^2 w_0(x,y)}{\partial y^2}\right)$$

$$= \frac{-EZ}{1-\mu^2}\left(\frac{\partial^2 w_0(x,y)}{\partial x^2} + \mu\,\frac{\partial^2 w_0(x,y)}{\partial y^2}\right) \tag{68}$$

Similarly,

$$\sigma_y = \frac{-EZ}{1-\mu^2}\left(\frac{\partial^2 w_0(x,y)}{\partial x^2} + \mu\,\frac{\partial^2 w_0(x,y)}{\partial y^2}\right) \tag{69}$$

Multiplying equations 17 and 68 gave

$$\sigma_x \in_x = \frac{EZ^2}{1 \quad \mu^2}\left(\left(\frac{\partial^2 w_0(x,y)}{\partial x^2}\right)^2 + \mu\,\frac{\partial^2 w_0(x,y)}{\partial x^2}\,.\,\frac{\partial^2 w_0(x,y)}{\partial y^2}\right) \tag{70}$$

Similarly,

$$\sigma_y \in_y = \frac{EZ^2}{1-\mu^2}\left(\left(\frac{\partial^2 w_0(x,y)}{\partial y^2}\right)^2 + \mu\,\frac{\partial^2 w_0(x,y)}{\partial x^2}\,.\,\frac{\partial^2 w_0(x,y)}{\partial y^2}\right) \tag{71}$$

Substituting equation 20 into equation 49 gave

$$\tau_{xy} = -\frac{EZ}{(1+\mu)}\cdot\frac{\partial^2 w_0(x,y)}{\partial x \partial y} \tag{72}$$

Multiplying equations 20 and 72 gave

$$\tau_{xy}\gamma_{xy} = \frac{2EZ^2}{(1+\mu)}\cdot\left(\frac{\partial^2 w_0(x,y)}{\partial x \partial y}\right)^2 \tag{73}$$

Integrating equation70 with respect to z in the range of $-\frac{h}{2} \leq Z \leq \frac{h}{2}$ gave

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \in_x = \left[\frac{EZ^3}{3(1-\mu^2)}\left(\left(\frac{\partial^2 w_0(x,y)}{\partial x^2}\right)^2 + \mu \frac{\partial^2 w_0}{\partial x^2} \cdot \frac{\partial^2 w_0}{\partial y^2}\right)\right]_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$= \frac{EZ^3}{3(1-\mu^2)} \cdot \left(\left[\frac{h}{2}\right]^3 - \left[\frac{-h}{2}\right]^3\right)\left(\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + \mu \frac{\partial^2 w_0}{\partial x^2} \cdot \frac{\partial^2 w_0}{\partial y^2}\right)$$

$$= \frac{Eh^3}{12(1-\mu^2)} \cdot \left(\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + \mu \frac{\partial^2 w_0}{\partial x^2} \cdot \frac{\partial^2 w_0}{\partial y^2}\right) \qquad (74)$$

Similarly

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y \in_y = \frac{Eh^3}{12(1-\mu^2)} \cdot \left(\left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + \mu \frac{\partial^2 w_0}{\partial x^2} \cdot \frac{\partial^2 w_0}{\partial y^2}\right) \qquad (75)$$

Integrating equation73 with respect to Z in the range of $-\frac{h}{2} \leq Z \leq \frac{h}{2}$ give

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy}\gamma_{yx} = \left[\frac{2EZ^3}{3(1+\mu)} \cdot \left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2\right]_{\frac{-h}{2}}^{\frac{h}{2}}$$

$$= \frac{2E}{3(1+u)} \cdot \left(\left[\frac{h}{2}\right]^3 - \left[\frac{-h}{2}\right]^3\right)\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2$$

$$= \frac{Eh^3}{6(1+\mu)} \cdot \left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 \qquad (76)$$

Multiplying equation 76 by $\frac{2(1-\mu)}{2(1-\mu)}$ gave

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy}\gamma_{yx} = \frac{2(1-\mu)Eh^3}{12(1-\mu^2)}.\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 \qquad (77)$$

$$Let \qquad D = \frac{Eh^3}{12(1-\mu^2)} \qquad (78)$$

Hence,

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \in_x = D.\left(\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + \mu \frac{\partial^2 w_0}{\partial x^2} . \frac{\partial^2 w_0}{\partial y^2}\right) \qquad (79)$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y \in_y = D.\left(\left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + \mu \frac{\partial^2 w_0}{\partial x^2} . \frac{\partial^2 w_0}{\partial y^2}\right) \qquad (80)$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy}\gamma_{xy} = 2(1-\mu)\, D.\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 \qquad (81)$$

Substituting equations 79, 80 and 81 into equation 45 gave

$$U = \frac{D}{2}\int_A \left[\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + \mu \frac{\partial^2 w_0}{\partial x^2} . \frac{\partial^2 w_0}{\partial y^2} + \left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + \mu \frac{\partial^2 w_0}{\partial x^2} . \frac{\partial^2 w_0}{\partial y^2} + 2(1-\mu) . \left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2\right] dA$$

$$= \frac{D}{2}\int \left[\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + 2\mu \frac{\partial^2 w_0}{\partial x^2} . \frac{\partial^2 w_0}{\partial y^2} + 2(1-\mu) . \left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2\right] dA$$

That is,

$$U = \frac{D}{2}\int\left[\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 w_0}{\partial x^2}.\frac{\partial^2 w_0}{\partial y^2} + 2(1-\mu).\left(\frac{\partial^2 w_0}{\partial x\partial y}\right)^2\right] dA \quad (82)$$

Since we have $w \approx w_0$ then equation 82 could be written as

$$U \approx \frac{D}{2}\int\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2} + 2(1-\mu).\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right] dA \quad (82a)$$

This means that equation 32 ≈ equation 32a. Equation 82 ≈ equation 82a.

Substituting equations 32 and 82 into equation 46 gave

$$\begin{aligned}\Pi = \frac{D}{2}\iint &[(w(x,y)^{IIx})^2 + (w(x,y)^{IIy})^2 + 2\mu\, w(x,y)^{IIx}\, w(x,y)^{IIy} \\ &+ 2(w(x,y)^{IIxy}) - 2\mu\,(w(x,y)^{IIxy})^2]\,\partial x\partial y \\ &- \frac{1}{2}\iint [P_x(w(x,y)^{Ix})^2 + P_y(w(x,y)^{Iy})^2 \\ &+ 2P_{xy}.w(x,y)^{Ix}.w(x,y)^{Iy}]\partial x\partial y \qquad (83)\end{aligned}$$

$$\begin{aligned}\Pi = \frac{D}{2}\iint &[(w''^x)^2 + (w''^y)^2 + 2\mu\, w''^x.w''^y + 2w''^{xy} - 2\mu\,(w''^{xy})^2]\,\partial x\partial y \\ &- \frac{1}{2}\iint [P_x(w'^x)^2 + P_y(w'^y)^2 + 2P_{xy}\ .\ w'^x.w(x,y)'^y]\partial x\partial y \qquad (83)\end{aligned}$$

Where

$$w^{Ix} = w(x,y)^{Ix} = \frac{\partial w(x,y)}{\partial x}$$

$$w^{Iy} = w(x,y)^{Iy} = \frac{\partial w(x,y)}{\partial y}$$

$$w^{IIx} = w(x,y)^{IIx} = \frac{\partial^2 w(x,y)}{\partial x^2}$$

$$w^{IIy} \ = \ w(x,y)^{IIy} = \frac{\partial^2 w(x,y)}{\partial y^2}$$

$$w^{IIxy} \ = \ w(x,y)^{IIxy} = \frac{\partial^2 w(x,y)}{\partial x \partial y}$$

$$Let \ \ R = \frac{x}{a} \quad and \ \ Q \ = \frac{y}{b} \ \ and \qquad (83A)$$

$$0 \leq R \leq 1, 0 \leq \ Q \leq \ 1 \ (R \ and \ Q \ are \ dimensionless \ quantities\ ) \ (83B)$$

Hence,

$$\Pi_x = \frac{Db}{2a^3} \iint \left[(w^{IIR})^2 + \frac{a^4}{b^4} \ (w^{IIQ})^2 + \ 2\frac{a^2}{b^2} \ (w^{IIRQ})^2 + 2\mu \frac{a^2}{b^2} w^{IIR} w^{IIQ} - \ 2\mu \frac{a^2}{b^2} \ (w^{IIRQ})^2 \right] \partial R \partial Q - \frac{bN_x}{2a} \iint (w^{IR})^2 \partial R \partial Q \qquad (84)$$

Where $\partial x \partial y = ab \partial R \partial Q$

$$\frac{\partial w}{\partial x} = \frac{1}{a} \frac{\partial w}{\partial R} \quad => \quad w^{IR} \ = \ a w^{Ix}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 w}{\partial R^2} \quad => \quad w^{IIR} \ = \ a^2 w^{IIx}$$

$$\frac{\partial w}{\partial y} = \frac{1}{b} \frac{\partial w}{\partial Q} \quad => \quad w^{IQ} \ = \ b w^{Iy}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{b^2} \frac{\partial w}{\partial Q^2} \quad => \quad w^{IIQ} \ = \ b^2 w^{IIy}$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{ab}\frac{\partial^2 w}{\partial R \partial Q} \quad => \quad w^{IIRQ} = abw^{IIxy}$$

$$(w^{IR})^2 = a^2(w^{Ix})^2$$

$$(w^{IIR})^2 = a^4(w^{IIx})^2$$

$$(w^{IQ})^2 = b^2(w^{Iy})^2$$

$$(w^{IIQ})^2 = b^2(w^{IIy})^2$$

$$(w^{IIRQ})^2 = a^2b^2(w^{IIxy})^2$$

$$\Pi_y = \frac{Da}{2b^3}\iint\left[\frac{b^4}{a^4}(w^{IIR})^2 + (w^{IIQ})^2 + 2\frac{b^2}{a^2}(w^{IIRQ})^2 + 2\mu\frac{b^2}{a^2}w^{IIR}w^{IIQ} - 2\mu\frac{b^2}{a^2}(w^{IIRQ})^2\right]\partial R\partial Q - \frac{aN_y}{2b}\iint(w^{IQ})^2\partial R\partial Q \qquad (85)$$

$$\Pi_{xy} = \frac{Db}{2a^2}\iint\left[(w^{IIR})^2 + \frac{a^4}{b^4}(w^{IIQ})^2 + 2\frac{a^2}{b^2}(w^{IIRQ})^2 + 2\mu\frac{a^2}{b^2}w^{IIR}w^{IIQ} - 2\mu\frac{a^2}{b^2}(w^{IIRQ})^2\right]\partial R\partial Q - \frac{bN_x}{2a}\iint 2w^{IR}w^{IQ}\partial R\partial Q \qquad (86)$$

Where

$$w'^R = w(R,Q)'^R = \frac{\partial w(R,Q)}{\partial R} \qquad (87)$$

$$w''^R = w(R,Q)''^R = \frac{\partial^2 w(R,Q)}{\partial R^2} \qquad (88)$$

$$w'^{Q} = w(R,Q)'^{Q} = \frac{\partial w(R,Q)}{\partial Q} \quad (89)$$

$$w''^{Q} = w(R,Q)''^{Q} = \frac{\partial^2 w(R,Q)}{\partial Q^2} \quad (90)$$

$$w''^{RQ} = w(R,Q)''^{RQ} = \frac{\partial^2 w(R,Q)}{\partial R \partial Q} \quad (91)$$

a = length of the edge of the plate along x direction.

b = length of the edge of the plate along y direction.

$$N_x - P_x \quad (92)$$

$$N_y = P_y \quad (93)$$

$$N_{xy} = P_{xy} \quad (94)$$

$\prod_x = total potential\ energy\ functional\ along\ x\ axis$

$\prod_y = total\ potential\ energy\ functional\ along\ y\ axis$

$\prod_{xy} = total\ potential\ energy\ functional\ along\ xy$

$$\partial x = a\partial R \quad (95)$$

$$\partial y = b\partial Q \quad (96)$$

When the displacement (at buckling) function, w is equal to the “exact shape function” of a thin rectangular flat plate, it can be shown that:

$$\int_0^1 \int_0^1 w^{IIR} . w^{IIQ} \, \partial R \partial Q \; = \int_0^1 \int_0^1 (w^{IIRQ})^2 \quad \partial R \partial Q \qquad (97)$$

See Ventsel and Krauthammer (2001). Take $w \; = A \; Sin \, \pi R \;\; Sin \, \pi Q,$ for example, as the exact shape function of a thin rectangular flat plate that is simple supported at all the four edges, then

$$w^{IIR} \; = -A \, \pi^2 Sin \pi R \; Sin \, \pi Q$$

$$w^{IIQ} \; = -A \, \pi^2 Sin \pi R \; Sin \, \pi Q$$

$$w^{IIRQ} \; = A \, \pi^2 Cos \pi R \; Cos \, \pi Q$$

$$w^{IIR} . w^{IIQ} \; = A^2 \pi^4 Sin^2 \pi R \; Sin^2 \, \pi Q$$

$$(w^{IIRQ})^2 \; = A^2 \pi^4 Cos^2 \pi R \; Cos^2 \, \pi Q$$

$$\int_0^1 \int_0^1 w^{IIR} . w^{IIQ} \, \partial R \partial Q \; = \frac{A^2 \pi^4}{4}$$

$$\int_0^1 \int_0^1 (w^{IIRQ})^2 \quad \partial R \partial Q \; = \frac{A^2 \pi^4}{4}$$

In the same manner, if the chosen displacement functions is a good approximation of the exact shape function, then

$$\int_0^1 \int_0^1 w^{IIR} . w^{IIQ} \, \partial R \; - \int_0^1 \int_0^1 (w^{IIRQ})^2 \, \partial R \partial Q \;\; \approx 0 \qquad (98)$$

Substituting equation(98) into equations (84), (85) and (86) respectively gave.

$$\Pi_x = \frac{Db}{2a^3}\int_0^1\int_0^1\left[(w^{IIR})^2 + \frac{a^4}{b^4}(w^{IIQ})^2 + \frac{2a^2}{b^2}(w^{IIRQ})^2\right]\partial R\partial Q - \frac{bN_x}{2a}\int_0^1\int_0^1 (w^{IR})^2\,\partial R\partial Q \qquad (99)$$

$$\Pi_x = \frac{D}{2b^2}\int_0^1\int_0^1\left[\frac{b^3}{a^3}(w^{IIR})^2 + \frac{a}{b}(w^{IIQ})^2 + \frac{2b}{a}(w^{IIRQ})^2\right]\partial R\partial Q \quad \frac{bN_x}{2a}\int_0^1\int_0^1 (w^{IR})^2\,\partial R\partial Q \qquad (99a)$$

$$\Pi_y = \frac{Da}{2b^3}\int_0^1\int_0^1\left[\frac{b^4}{a^4}(w^{IIR})^2 + (w^{IIQ})^2 + \frac{2b^2}{a^2}(w^{IIRQ})^2\right]\partial R\partial Q - \frac{aN_x}{2b}\int_0^1\int_0^1 (w^{IQ})^2\,\partial R\partial Q \qquad (100)$$

$$\Pi_y = \frac{D}{2b^2}\int_0^1\int_0^1\left[\frac{b^3}{a^3}(w^{IIR})^2 + \frac{a}{b}(w^{IIQ})^2 + \frac{2b}{a}(w^{IIRQ})^2\right]\partial R\partial Q - \frac{aN_y}{2b}\int_0^1\int_0^1 (w^{IQ})^2\,\partial R\partial Q \qquad (100a)$$

$$\Pi_{xy} = \frac{Db}{2a^3}\int_0^1\int_0^1\left[(w^{IIR})^2 + \frac{a^4}{b^4}(w^{IIQ})^2 + \frac{2a^2}{b^2}(w^{IIRQ})^2\right]\partial R\partial Q - \frac{bN_{xy}}{2a}\int_0^1\int_0^1 (w^{IR})^2 . w^{IQ}\,\partial R\partial Q \qquad (101)$$

$$\Pi_{xy} = \frac{D}{2b^2}\int_0^1\int_0^1\left[\frac{b^3}{a^3}(w^{IIR})^2 + \frac{a}{b}(w^{IIQ})^2 + \frac{2b}{a}(w^{IIRQ})^2\right]\partial R\partial Q - \frac{bN_{xy}}{2a}\int_0^1\int_0^1 (w^{IR})^2.w^{IQ}\,\partial R\partial Q \tag{101a}$$

3.2 DISPLACEMENT FUNCTION

It can be seen that the potential energy functional of a thin rectangular flat plate under in-plane stress is dependent on the displacement function, w in the z direction. Let this displacement function, w be continuous and differentiable. Expanding the function in Taylor-Mclaurin series gave.

$$w = w(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{F^{(m)}(x_0).F^{(n)}(y_0)}{m!\,n!}(x-x_0)^m.(y-y_0)^n$$

$F^{(m)}(x_0)$ is the mth partial derivative of the function

to x and $F^{(n)}(y_0)$ is the nth partial derivative of the function w(x,y) with respect to y. m! and n! are factorials of m and n respectively. x_0 and y_0 are the points at the origin. In this case, the origin was taken to be zero. Therefore, $x_0 = y_0 = 0$.

Thus, the Taylor - Mclaurrin expression above became

$$w = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{F^{(m)}(0).F^{(n)}(0)}{m!\,n!}x^m.y^n$$

$$Let \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{F^{(m)}(0).F^{(n)}(0)}{m!\,n!},$$

*be some unknown constants*And be represented as I_m J_n, therefore, the infinite series became

$$w = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} I_m J_n \; x^m . y^n \qquad (102)$$

Truncating this infinite series (equation 102) at a point m = n = 4 gave a finite series that could approximate the shape function as:

$$w = \sum_{m=0}^{4}\sum_{n=0}^{4} I_m J_n \; x^m . y^n \qquad (103)$$

$But\ R = \frac{x}{a} \quad and \quad Q = \frac{y}{b}\ .\ That\ is$

$x = aR \quad and \quad y = bQ$. Substituting these into equation (103) gave

$$w - \sum_{m=0}^{4}\sum_{n=0}^{4} I_m J_n \; (aR)^m . (bQ)^n$$

$$= \sum_{m=0}^{4}\sum_{n=0}^{4} I_m J_n \; a^m b^n R^m Q^n$$

$Let\ \ a_m - I_m . a^m \quad and\ \ b_n = J_m . b^n,\ therefore,$

$$w = \sum_{m=0}^{4}\sum_{n=0}^{4} a_m b_n \; R^m . Q^n \qquad (104)$$

Expanding equation (104) up to m = 4 gave

$$w = \sum_{n=0}^{4} b_n Q^n (a_0 + a_1 R + a_2 R^2 + a_3 R^3 + a_4 R^4) \qquad (105)$$

Expanding equation (105) up to n = 4 gave

$$w = (a_0 + a_1 R + a_2 R^2 + a_3 R^3 + a_4 R^4)\,(b_0 + b_1 Q + b_2 Q^2 + b_3 Q^3 + b_4 Q^4) \qquad (106)$$

That is

$$\begin{aligned} w = \; & a_0 b_0 + a_0 b_1 Q + a_0 b_2 Q^2 + a_0 b_3 Q^3 + a_0 b_4 Q^4 + \\ & a_1 b_0 R + a_1 b_1 RQ + a_1 b_2 RQ^2 + a_1 b_3 RQ^3 + a_1 b_4 R Q^4 + \\ & a_2 b_0 R^2 + a_2 b_1 R^2 Q + a_2 b_2 R^2 Q^2 + a_2 b_3 R^2 Q^3 + a_2 b_4 R^2 Q^4 + \\ & a_3 b_0 R^3 + a_3 b_1 R^3 Q + a_3 b_2 R^3 Q^2 + a_3 b_3 R^3 Q^3 + a_3 b_4 R^3 Q^4 + \\ & a_4 b_0 R^4 + a_4 b_1 R^4 Q + a_4 b_2 R^4 Q^2 + a_4 b_3 R^4 Q^3 + a_4 b_4 R^4 Q^4 \qquad (107) \end{aligned}$$

Similarly, truncating equation 102 at M = N = 5 gave

$$w = \sum_{m-0}^{5} \sum_{n-0}^{5} \mathrm{a_m}\, \mathrm{b_n}\; R^m . Q^n$$

Expanding this equation gave

$$\begin{aligned} w = \; & (a_0 + a_1 R + a_2 R^2 + a_3 R^3 + a_4 R^4 + a_5 R^5) * \\ & (b_0 + b_1 Q + b_2 Q^2 + b_3 Q^3 + b_4 Q^4 + b_5 Q^5) \qquad (107a) \end{aligned}$$

Thus, eq(107) was a 25-term finite series that was used in approximating the infinite series of eq(102), which was the exact shape function for buckled shape of a rectangular thin plate under in-plane loading. This is polynomial approximation of the shape function of a thin rectangular flat plate. Substitution of boundary conditions into it gave the polynomial shape functions that best defined the buckled shapes of the arbitrary thin rectangular flat plates. Equation 107a was also used to approximate the infinite series. However, equation 107a was used mainly to see whether the function converged at M = N = 4 or not.

A thin rectangular flat plate with edge numbers was as shown in Fig. 3.2. There were three conditions considered. They were simply supported (S), clamped (C) and free (F). SSSS represented a plate with four edges simply supported. CCCC represented a plate with four edges clamped. SCSC represented a plate with edges 1 and 3 simply supported and edges 2 and 4 clamped. The following cases of different edge conditions were considered:

SSSS

CCCC

CSCS

CSSS

CCSC

CCSS

CSFS

SSFS

CCFC

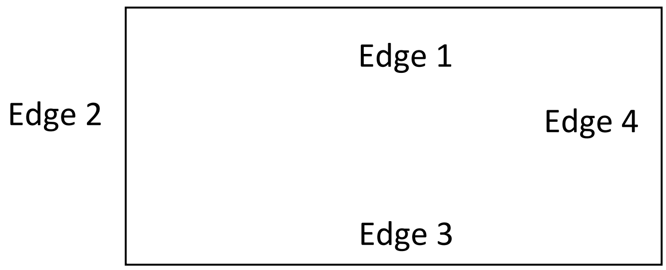

Figure 3.2: Thin rectangular flat with edge numbers

3.2.1 DISPLACEMENT FUNCTION FOR SSSS PLATE

The boundary conditions for SSSS plate were

$$w(R-0) - w^{IIR}(R-0) - 0$$

$$w(R=1) = w^{IIR}(R=1) = 0$$

$$w(Q=0) = w^{IIQ}(Q=0) = 0$$

$$w(Q=1) = w^{IIQ}(Q=1) = 0$$

For M = N = 4

Substituting $w(R=0)=0$ into equation(107) gave

$$a_0 = 0$$

Substituting $w^{IIR}(R=0) = 0$ into equation(107) gave

$$a_2 = 0$$

Similarly, substituting $w(Q=0)=0$ and $w^{IIQ}(Q=0)= 0$ into equation(107) respectively gave

$b_0 = 0$

$b_2 = 0$

Substituting $w(R=1)=0$ and $w^{IIR}(R=1)= 0$ into equation(107) respectively gave

$a_1 + a_3 + a_4 = 0 \ and$

$6a_3 + 12a_4 = 0$

Solving the two equations simultaneously gave

$a_1 = a_4$

$a_3 = -2a_4$

Similarly, substituting $w(Q=1)=0$ and $w^{IIQ}(Q=1)= 0$ into equation(107) gave

$b_1 = b_4$

$b_3 = -2b_4$

Substituting the values of $a_0, \ a_1, a_2, \ a_3, \ a_4, \ b_0, \ b_1, b_2, \ b_3 \ and \ b_4$ in to equation(107) gave

$$w = (R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)a_4b_4$$

$$w = A(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4) \qquad (108)$$

$Where\ A = a_4b_4$

3.2.2 DISPLACEMENT FUNCTION FOR CCCC PLATE

The boundary conditions for CCCC plate were

$$w(R = 0) = w^{IR}(R = 0) = 0$$

$$w(R = 1) = w^{IR}(R = 1) = 0$$

$$w(Q = 0) = w^{IQ}(Q = 0) = 0$$

$$w(Q = 1) = w^{IQ}(Q = 1) = 0$$

For M = N = 4

Substituting

$w(R = 0) = 0, w^{IR}(R = 0) = 0, w(Q = 0) = 0\ and\ w^{IQ}(Q = 0) = 0$ into

equation(107) one after the other respectively gave

$$a_0 = 0$$

$$a_1 = 0$$

$$b_0 = 0$$

$$b_1 = 0$$

Substituting $w(Q = 1) = 0 \; and \; W^{IQ}(Q = 1) = 0$ into equation(107) one after the other respectively gave

$$a_2 + a_3 + a_4 = 0$$

$$2a_2 + 3a_3 + 4a_4 = 0$$

Solving the two equations simultaneously gave

$$a_2 = a_4$$

$$a_3 = -2a_4$$

Similarly, substituting $w(Q = 1) = 0 \; and \; W^{IQ}(Q = 1) = 0$, one after the other into equation(107) and solving the resulting two equations simultaneously gave

$$b_2 = b_4$$

$$b_3 = -2b_4$$

Substituting $a_0, \; a_1, a_2, \; a_3, \; a_4, \; b_0, \; b_1, b_2, \; b_3 \; and \; b_4$ in to equation(107) gave

$$w = A(R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \qquad (109)$$

3.2.3 DISPLACEMENT FUNCTION FOR CSCS PLATE

The boundary conditions for CSCS plate are

$$w(R = 0) = w^{IIR}(R = 0) = 0$$

$w(R = 1) = w^{IIR}(R = 1) = 0$

$w(Q = 0) = w^{IQ}(Q = 0) = 0$

$w(Q = 1) = w^{IQ}(Q = 1) = 0$

For M = N = 4

Substituting

$w(R = 0) = 0, w^{IIR}(R = 0) = 0,\ w(Q = 0) = 0\ and\ W^{IQ}(Q = 0) = 0$ into equation(107) one after the other respectively gave

$a_0 - 0$

$a_2 = 0$

$b_0 = 0$

$b_1 = 0$

Substituting $w(R = 1) = 0\ and\ w^{IIR}(R = 1) = 0$ one after the other gave, respectively, the two equations below

$a_1 + a_3 + a_4 = 0$

$6a_3 + 12a_4 = 0$

On solving simultaneously, the two equations gave

$a_1 = a_4$

$a_3 = -2a_4$

Similarly, substituting $w(Q = 1) = 0 \; and \; w^{IQ}(Q = 1) = 0$ into equation (107) one after the other and solving the resulting two equations simultaneously gave

$b_2 = b_4$

$b_3 = -2b_4$

Substituting these parameters into equation(107) gave

$$w - A(R - 2R^3 + R^4)\,(Q^2 - 2Q^3 + Q^4) \qquad (110)$$

For M = N = 5

Substituting

$w(R = 0) = 0, w^{IIR}(R = 0) = 0, \; w(Q = 0) = 0 \; and \; W^{IQ}(Q = 0) = 0$ into equation(107a) one after the other respectively gave

$a_0 = 0 \; ; \; a_2 = 0 \; ; \; b_0 = 0 \; ; \; b_1 = 0$

Substituting $w(R = 1) = 0 \; and \; w^{IIR}(R = 1) = 0$ one after the other into equation(107a) gave, respectively, the two equations below

$a_1 + a_3 + a_4 + a_5 = 0$

$6a_3 + 12a_4 + 20a_5 = 0$

On solving simultaneously, the two equations gave

$a_1 = a_4 + 2.33a_5$

$a_3 = -2a_4 - 3.33a_5$

Similarly, substituting $w(Q = 1) = 0 \; and \; w^{IQ}(Q = 1) = 0$ into equation (107a) one after the other and solving the resulting two equations simultaneously gave

$b_2 = b_4 + 2b_5$

$b_3 = -2b_4 - 3b_5$

Substituting these parameters into equation(107a) gave

$$\begin{aligned} w = {} & A1(R - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \\ & + A2(R - 2R^3 + R^4)(2Q^2 - 3Q^3 + Q^5) \\ & + A3(2.33R - 3.33R^3 + R^5)(Q^2 - 2Q^3 + Q^4) \\ & + A4(2.33R - 3.33R^3 + R^5)(2Q^2 - 3Q^3 + Q^5) \quad (110a) \end{aligned}$$

$Where \; A1 = a_4 b_4 \; ; A2 = a_4 b_5; A3 = a_5 b_4; A4 = a_5 b_5$

3.2.4 DISPLACEMENT FUNCTION FOR CSSS PLATE

The boundary conditions for CSSS plate were

$w(R = 0) = w^{IIR}(R = 0) = 0$

$w(R = 1) = w^{IIR}(R = 1) = 0$

$w(Q = 0) = w^{IQ}(Q = 0) = 0$

$w(Q = 1) = w^{IIQ}(Q = 1) = 0$

For M = N = 4

Substituting

$w(R=0)=\ 0, w^{IIR}(R=0)=\ 0,\ w(Q=0)=\ 0\ \ and\ \ w^{IQ}(Q=0)=\ 0$ into equation(107) one after the other respectively gave

$a_0=0\ ;\ a_2=0\ ;\ b_0=0\ ;\ b_1=0$

Substituting $w(R=1)=0\ and\ \ w^{IIR}(R=1)=\ 0$ into equation(107) one after the other gave, respectively, the two equations below

$a_1+\ a_3+\ a_4=0$

$6a_3+\ 12a_4=0$

On solving simultaneously, gave

$a_1=a_4$

$a_3=\ -2a_4$

Similarly, substituting $w(Q\ =1)=\ 0\ and\ \ w^{IIQ}(Q=1)=\ 0$ into equation (107) one after the other and solving the resulting two equations simultaneously gave

$b_2=1.5b_4$

$b_3=\quad 2.5b_4$

Substituting these parameters into equation(107) gave

$$w = A(R - 2R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4) \qquad (111)$$

3.2.5 DISPLACEMENT FUNCTION FOR CCSC PLATE

The boundary conditions for CCSC plate were

$$w(R = 0) = w^{IR}(R = 0) = 0$$

$$w(R - 1) - w^{IR}(R - 1) - 0$$

$$w(Q = 0) = w^{IQ}(Q = 0) = 0$$

$$w(Q = 1) = W^{IIQ}(Q = 1) = 0$$

For M = N = 4

Substituting

$w(R = 0) = 0, w^{IR}(R = 0) = 0,\ w(Q = 0) = 0\ and\ W^{IQ}(Q = 0) = 0$ one

after the other into equation(107) respectively gave

$$a_0 = 0$$

$$a_1 = 0$$

$$b_0 = 0$$

$$b_1 = 0$$

Substituting $w(R=1)=0$ and $w^{IR}(R=1)=0$ one after the other into equation(107) gave the two equations respectively:

$$a_2 + a_3 + a_4 = 0$$

$$2a_2 + 3a_3 + 4a_4 = 0$$

Solving the two equation simultaneously gave

$$a_2 = a_4$$

$$a_3 = -2a_4$$

Similarly, substituting $w(Q=1)=0$ and $w^{IIQ}(Q=1)=0$ one after the other into equation (107) and solving the resulting two equations simultaneously gave

$$b_2 = 1.5b_4$$

$$b_3 = -2.5b_4$$

Substituting these parameters into equation(107) gave

$$w = A(R^2 - 2R^3 + R^4)\,(1.5R^2 - 2.5R^3 + R^4) \qquad (112)$$

3.2.6 DISPLACEMENT FUNCTION FOR CCSS PLATE

The boundary conditions for CCSS plate were

$$w(R=0) = w^{IR}(R=0) = 0$$

$$w(R=1) = w^{IIR}(R=1) = 0$$

$w(Q = 0) = w^{IQ}(Q = 0) = 0$

$w(Q = 1) = w^{IIQ}(Q = 1) = 0$

For M = N = 4

Substituting

$w(R = 0) = 0, w^{IR}(R = 0) = 0,\ w(Q = 0) = 0\ and\ w^{IQ}(Q = 0) = 0$ one after the other into equation(107) gave

$a_0 = 0\ ;\ a_1 = 0\ ;\ b_0 = 0\ ;\ b_1 = 0$

Substituting $w(R - 1) - 0\ and\ w^{IIR}(R - 1) - 0$ one after the other into equation(107) gave the two equations respectively

$a_2 + a_3 + a_4 = 0$

$2a_2 + 6a_3 + 12a_4 = 0$

Solving the two equations simultaneously gave

$a_1 = 1.5a_4$

$a_3 = -2.5a_4$

Similarly, substituting $w(Q = 1) = 0\ and\ w^{IIQ}(Q = 1) = 0$ one after the other into equation (107) and solving the resulting two equations simultaneously gave

$b_2 = 1.5b_4$

$b_3 = -2.5b_4$

Substituting these parameters into equation(107) gave

$$w = A(1.5R^2 - 2.5R^3 + R^4)\,(1.5Q^2 - 2.5Q^3 + Q^4) \qquad (113)$$

For M = N = 5

Substituting

$w(R = 0) = 0, w^{IR}(R = 0) = 0,\ w(Q = 0) = 0\ and\ w^{IQ}(Q = 0) = 0$ one after the other into equation(107a) gave

$a_0 = 0\ ;\ a_1 = 0\ ;\ b_0 = 0\ ;\ b_1 = 0$

Substituting $w(R = 1) = 0\ and\ w^{IIR}(R = 1) = 0$ one after the other into equation(107a) gave the two equations respectively

$a_2 + a_3 + a_4 + a_5 = 0$

$2a_2 + 6a_3 + 12a_4 + 20a_5 = 0$

Solving the two equations simultaneously gave

$a_2 = 1.5a_4 + 3.5a_5$

$a_3 = -2.5a_4 - 4.5a_5$

Similarly, substituting $w(Q = 1) = 0\ and\ w^{IIQ}(Q = 1) = 0$ one after the other into equation (107a) and solving the resulting two equations simultaneously gave

$b_2 = 1.5b_4 + 3.5b_5$

$b_3 = -2.5b_4 - 4.5b_5$

Substituting these parameters into equation(107a) gave

$$\begin{aligned} w = {} & A1(1.5R^2 - 2.5R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4) \\ & + A2(1.5R^2 - 2.5R^3 + R^4)(3.5Q^2 - 4.5Q^3 + Q^5) \\ & + A3(3.5R^2 - 4.5R^3 + R^5)(1.5Q^2 - 2.5Q^3 + Q^4) \\ & + A4(3.5R^2 - 4.5R^3 + R^5)(3.5Q^2 - 4.5Q^3 + Q^5) \quad (113a) \end{aligned}$$

3.2.7 DISPLACEMENT FUNCTION FOR CSFS PLATE

The boundary conditions for CSFS plate were

$w(R = 0) = w^{IIR}(R = 0) = 0$

$w(R = 1) = w^{IIR}(R = 1) = 0$

$w(Q = 0) = w^{IQ}(Q = 0) = 0$

$M_y(Q = 1) = V_y(Q = 1) = 0$

For M = N = 4

Substituting

$w(R = 0) = 0, w^{IIR}(R = 0) = 0, \; w(Q = 0) = 0 \; and \; w^{IQ}(Q = 0) = 0$ one

after the other into equation(107) gave

$a_0 = 0$

$a_2 = 0$

$b_0 = 0$

$b_1 = 0$

Substituting $w(R = 1) = 0 \; and \; W^{IIR}(R = 1) = 0$ one after the other into equation(107) and solving the resulting two equations respectively gave

$a_1 = a_4$

$a_3 = -2a_4$

The condition, $M_y(Q = 1) = 0 \; and \; V_y(Q = 1) = 0$ is too complex to analyse. Hence, substituting the already known parameter into equation(107) gave

$w = a_4(R - 2R^3 + R^4)(b_2Q^2 + b_3Q^3 + b_4Q^4)$

Assuming $c_2 - a_4b_2$

$c_3 = a_4b_3$

$c_4 = a_4b_4 \; then$

$$w = (R - 2R^3 + R^4)(C_2Q^2 + C_3Q^3 + C_4Q^4) \qquad (114)$$

Equation(114) was the approximate polynomial shape function with three unknown parameters for CSFS plate.

3.2.8 DISPLACEMENT FUNCTION FOR SSFS PLATE

The boundary conditions for SSFS plate were

$$w(R=0) = W^{IIR}(R=0) = 0$$

$$w(R=1) = W^{IIR}(R=1) = 0$$

$$w(Q=0) = W^{IIQ}(Q=0) = 0$$

$$M_y(Q=1) = V_y(Q=1) = 0$$

Substituting these parameters into equation(107) gave

$$w = (R - 2R^3 + R^4)(C_1Q + C_3Q^3 + C_4Q^4) \quad (115)$$

For M = N = 4

Substituting

$w(R=0) = 0, w^{IIR}(R=0) = 0,\ w(Q=0) = 0\ and\ w^{IIQ}(Q=0) = 0$ one after the other into equation(107) gave

$$a_0 = 0$$

$$a_2 = 0$$

$$b_0 = 0$$

$$b_2 = 0$$

Substituting $w(R=1) = 0\ and\ w^{IIR}(R=1) = 0$ one after the other into equation(107) and solving the resulting two equations simultaneously gave

$a_1 = a_4$

$a_3 = -2a_4$

3.2.9 DISPLACEMENT FUNCTION FOR CCFC PLATE

The boundary conditions for CCFC plate were

$w(R = 0) = W^{IR}(R = 0) = 0$

$w(R = 1) = W^{IR}(R = 1) = 0$

$w(Q = 0) = W^{IQ}(Q = 0) = 0$

$M_y(Q = 1) = V_y(Q = 1) = 0$

For M = N = 4

Substituting

$w(R = 0) = 0, w^{IR}(R = 0) = 0, w(Q = 0) = 0$ and $w^{IQ}(Q = 0) = 0$ one

after the other into equation(107) gave

$a_0 = 0$

$a_1 = 0$

$b_0 = 0$

$b_1 = 0$

Substituting $w(R=1)=0\ and\ w^{IR}(R=1)=\ 0$ one after the other into equation(107) and solving the resulting two equations respectively gave

$$a_2 = a_4$$

$$a_3 = -2a_4$$

Substituting these parameters into equation(107) gave

$$w = (R^2 - 2R^3 + R^4)\,(C_2Q^2 + C_3Q^3 + C_4Q^4) \qquad (117)$$

3.3 TOTAL POTENTIAL ENERGY FUNCTIONAL FOR SSSS PLATE

For M = N = 4

Differentiating equation (108) partially with respect to either R or Q or both gave the following equation:

$$w^{IR} = A(1 - 6R^2 + 4R^3)\,(Q - 2Q^3 + Q^4)$$

$$w^{IIR} = A(12R^2 - 12R)\,(Q - 2Q^3 + Q^4)$$

$$w^{IQ} = A(R^2 - 2R^3 + R^4)\,(1 - 6Q^2 + 4Q^3)$$

$$w^{IIQ} = A(R^2 - 2R^3 + R^4)\,(12Q^2 - 12Q)$$

$$w^{IIRQ} = A(1 - 6R^2 + 4R^3)\,(1 - 6Q^2 + 4Q^3)$$

Squaring each of the above equations gave the following equations respectively:

$$(w^{IR})^2 = A^2(1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)(Q^2 - 4Q^4 + 2Q^5 + 4Q^6 - 4Q^7 + Q^8)$$

$$(w^{IIR})^2 = A^2(144R^4 - 288R^3 + 144R^2)(Q^2 - 4Q^4 + 2Q^5 + 4Q^6 - 4Q^7 + Q^8)$$

$$(w^{IQ})^2 = A^2(R^2 - 4R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)(1 - 12Q^2 + 8Q^3 + 36Q^4 - 48Q^5 + 16Q^6)$$

$$(w^{IIQ})^2 = A^2(R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)(144Q^4 - 288Q^3 + 144Q^2)$$

$$(w^{IIRQ})^2 = A^2(1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)(1 - 12Q^2 + 8Q^3 + 36Q^4 - 48Q^5 + 16Q^6)$$

Integrating these five squared equations partially with respect to R and Q in a closed domain respectively gave:

$$\int_0^1 \int_0^1 (w^{IR})^2 \partial R \partial Q = A^2(0.48571)(0.04921) = 0.0239A^2$$

$$\int_0^1 \int_0^1 (w^{IIR})^2 \partial R \partial Q = A^2(4.8)(0.04921) = 0.23621A^2$$

$$\int_0^1 \int_0^1 (w^{IQ})^2 \partial R \partial Q = A^2(0.04921)(0.48571) = 0.0239A^2$$

$$\int_0^1 \int_0^1 (w^{IIQ})^2 \partial R \partial Q = A^2(0.04921)(4.8) = 0.23621A^2$$

$$\int_0^1 \int_0^1 (w^{IIRQ})^2 \partial R \partial Q = A^2(0.48571)(0.48571) = 0.23591A^2$$

Substituting the above integrals into equations (99) and (100) gave

$$\prod_x = \frac{DA^2}{2Pa^2}(0.23621 + 0.23621P^4 + 0.47182P^2) - \frac{N_x A^2}{2P}(0.0239) \quad (118)$$

$$\prod_y = \frac{DPA^2}{2b^2}\left(\frac{0.23621}{P^4} + 0.23621 + \frac{0.47182}{P^2}\right) - \frac{N_y PA^2}{2}(0.0239) \quad (119)$$

$$P = \frac{a}{b} \quad (120)$$

3.4 TOTAL POTENTIAL ENERGY FUNCTIONAL FOR CCCC PLATE

For M = N = 4

Differentiating equation(109) partially with respect to either R or Q or both gave the following equations:

$$w^{IR} = A(2R - 6R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4)$$

$$w^{IIR} = A(2 - 12R + 12R^2)(Q^2 - 2Q^3 + Q^4)$$

$$w^{IQ} = A(R^2 - 2R^3 + R^4)(2Q - 6Q^2 + 4Q^3)$$

$$w^{IIQ} = A(R^2 - 2R^3 + R^4)(2 - 12Q + 12Q^2)$$

$$w^{IIRQ} = A(2R - 6R^2 + 4R^3)(2Q - 6Q^2 + 4Q^3)$$

Squaring each of the above five equations respectively gave

$$(w^{IR})^2 = A^2(4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6)(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$$

$$(w^{IIR})^2 = A^2(4 - 48R + 192R^2 - 288R^3 + 144R^4)(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$$

$$(w^{IQ})^2 = A^2(R^4 - 4R^5 + 6R^6 - 4R^7 + R^8)\,(4Q^2 - 24Q^3 + 52Q^4 - 48Q^5 + 16Q^6)$$

$$(w^{IIQ})^2 = A^2(R^4 - 4R^5 + 6R^6 - 4R^7 + R^8)(4 - 48Q + 192Q^2 - 288Q^3 + 144Q^2)$$

$$(W^{IIRQ})^2 = A^2(4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6)(4Q^2 - 24Q^3 + 52Q^4 - 48Q^5 + 16Q^6)$$

Integrating these five squared equations partially with respect to R and Q in a closed domain respectively gave:

$$\int_0^1 \int_0^1 (w^{IR})^2 \partial R \partial Q = A^2(0.01905)(0.00159) = 0.00003A^2$$

$$\int_0^1 \int_0^1 (w^{IIR})^2 \partial R \partial Q = A^2(0.8)(0.00159) = 0.00127A^2$$

$$\int_0^1 \int_0^1 (w^{IQ})^2 \partial R \partial Q = A^2(0.00159)(0.01905) = 0.00003A^2$$

$$\int_0^1 \int_0^1 (w^{IIQ})^2 \partial R \partial Q = A^2(0.00159)(0.8) = 0.00127A^2$$

$$\int_0^1 \int_0^1 (w^{IIRQ})^2 \partial R \partial Q = A^2(0.01905)(0.01905) = 0.00036A^2$$

Substituting the above integrals into equations (99) and (100) gave

$$\Pi_x = \frac{DA^2}{2P\alpha^2}(0.00127 + 0.00127P^4 + 0.00073P^2) - \frac{N_x A^2}{2P}(0.00003) \quad (121)$$

$$\prod_y = \frac{DPA^2}{2b^2}\left(\frac{0.00127}{P^4} + 0.00127 + \frac{0.00073}{P^2}\right) - \frac{N_y PA^2}{2}(0.00003) \qquad (122)$$

3.5 TOTAL POTENTIAL ENERGY FUNCTIONAL FOR CSCS PLATE

$$\int_0^1 \int_0^1 (w^{IR})^2 \partial R \partial Q = A^2(0.48571)(0.00159) = 0.00077A^2$$

$$\int_0^1 \int_0^1 (w^{IIR})^2 \partial R \partial Q = A^2(4.8)(0.00159) = 0.00763A^2$$

$$\int_0^1 \int_0^1 (w^{IQ})^2 \partial R \partial Q = A^2(0.04921)(0.01905) = 0.00094A^2$$

$$\int_0^1 \int_0^1 (w^{IIQ})^2 \partial R \partial Q = A^2(0.04921)(0.8) = 0.03937A^2$$

$$\int_0^1 \int_0^1 (w^{IIRQ})^2 \partial R \partial Q = A^2(0.48571)(0.01905) = 0.00925A^2$$

Substituting the above integrals into equations (99) and (100) gave

$$\prod_x = \frac{DA^2}{2Pa^2}(0.00763 + 0.03937P^4 + 0.0185P^2) - \frac{N_x A^2}{2P}(0.00077) \qquad (123a)$$

$$\prod_y = \frac{DPA^2}{2b^2}\left(\frac{0.00763}{P^4} + 0.03937 + \frac{0.0185}{P^2}\right) - \frac{N_y PA^2}{2}(0.00094) \qquad (124)$$

For M = N = 5

Differentiating equation(110a) partially with respect to either R or Q or both gave the following equations

$$w^{IR} = A_1(1-6R^2+4R^3)(Q^2-2Q^3+Q^4) + A_2(1-6R^2+4R^3)(2Q^2-3Q^3+Q^5) + A_3(2.33-10R^2+5R^4)(Q^2-2Q^3+Q^4) + A_4(2.33-10R^2+5R^4)(2Q^2-3Q^3+Q^5)$$

$$w^{IIR} = A_1(12R^2-12R)(Q^2-2Q^3+Q^4) + A_2(12R^2-12R)(2Q^2-3Q^3+Q^5) + A_3(20R^3-20R^2)(Q^2-2Q^3+Q^4) + A_4(20R^3-20R^2)(2Q^2-3Q^3+Q^5)$$

$$w^{IQ} = A_1(R-2R^3+R^4)(2Q-6Q^2+4Q^3) + A_2(R-2R^3+R^4)(4Q-9Q^2+5Q^4) + A_3(2.33R-3.33R^3+R^5)(2Q-6Q^2+4Q^3) + A_4(2.33R-3.33R^3+R^5)(4Q-9Q^2+5Q^4)$$

$$w^{IIQ} = A_1(R-2R^3+R^4)(2-12Q+12Q^2) + A_2(R-2R^3+R^4)(4-18Q+20Q^3) + A_3(2.33R-3.33R^3+R^5)(2-12Q+12Q^2) + A_4(2.33R-3.33R^3+R^5)(4-18Q+20Q^3)$$

$$w^{IIRQ} = A_1(1-6R^2+4R^3)(2Q-6Q^2+4Q^3) + A_2(1-6R^2+4R^3)(4Q-9Q^2+5Q^4) + A_3(2.33-10R^2+5R^4)(2Q-6Q^2+4Q^3) + A_4(2.33-10R^2+5R^4)(4Q-9Q^2+5Q^4)$$

Squaring each of the above five equations gave the equations respectively

$$(w^{IR})^2$$
$$= A_1^2(1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)\,(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$$
$$+ A_2^2(1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)\,(4Q^4 - 12Q^5 + 9Q^5 + 4Q^7 - 6Q^8 + Q^{10})$$
$$+ A_3^2(5.44 - 46.667R^2 + 123.33R^4 - 100R^6 + 25R^8)\,(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$$
$$+ A_4^2(5.44 - 46.667R^2 + 123.33R^4 - 100R^6 + 25R^8)\,(4Q^4 - 12Q^5 + 9Q^6 + 4Q^7 - 6Q^8 + Q^{10})$$

$$+2A_1A_2(1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)\,(2Q^4 - 7Q^5 + 8Q^6 - 2Q^7 - 2Q^8 + Q^9)$$
$$+ 2A_1A_3(2.33 - 24R^2 + 9.33R^3 + 65R^4 - 40R^5 - 30R^6 + 20R^7)\,(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$$
$$+ 2A_1A_4(2.33 - 24R^2 + 9.33R^3 + 65R^4 - 40R^5 - 30R^6 + 20R^7)\,(2Q^4 - 7Q^5 + 8Q^6 - 2Q^7 - 2Q^8 + Q^9)$$

$$+2A_2A_3(2.33 - 24R^2 + 9.33R^3 + 65R^4 - 40R^5 - 30R^6 + 20R^7)\,(2Q^4 - 7Q^5 + 8Q^6 - 2Q^7 - 2Q^8 + Q^9)$$
$$+ 2A_2A_4(2.33 - 24R^2 + 9.33R^3 + 65R^4 - 40R^5 - 30R^6 + 20R^7)\,(4Q^4 - 12Q^5 + 9Q^6 + 4Q^7 - 6Q^8 + Q^{10})$$
$$+ 2A_3A_4(5.44 - 46.667R^2 + 123.33R^4 - 100R^6 + 25R^8)\,(2Q^4 - 7Q^5 + 8Q^6 - 2Q^7 - 2Q^8 + Q^9)$$

$$(w^{IIR})^2$$
$$= A_1^2(144R^2 - 288R^3 + 144R^4)\,(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$$
$$+ A_2^2(144R^2 - 288R^3 + 144R^4)\,(4Q^4 - 12Q^5 + 9Q^6 + 4Q^7 - 6Q^8 + Q^{10})$$
$$+ A_3^2(400R^2 - 800 - 100R^6 + 400R^6)\,(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$$
$$+ A_4^2(400R^2 - 800 - 100R^6 + 400R^6)\,(4Q^4 - 12Q^5 + 9Q^6 + 4Q^7 - 6Q^8 + Q^{10})$$

$$+2A_1A_2(144R^2 - 288R^3 + 144R^4)\,(2Q^4 - 7Q^5 + 8Q^6 - 2Q^7 - 2Q^8 + Q^9)$$
$$+ 2A_1A_3(240R^2 - 240R^3 - 240R^4 + 240R^5)\,(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$$
$$+ 2A_1A_4(240R^2 - 240R^3 - 240R^4 + 240R^5)\,(2Q^4 - 7Q^5 + 8Q^6 - 2Q^7 - 2Q^8 + Q^9)$$

$$+2A_2A_3(240R^2 - 240R^3 - 240R^4 + 240R^5)\,(2Q^4 - 7Q^5 + 8Q^6 - 2Q^7 - 2Q^8 + Q^9)$$
$$+ 2A_2A_4(240R^2 - 240R^3 - 240R^4 + 240R^5)\,(4Q^4 - 12Q^5 + 9Q^6 + 4Q^7 - 6Q^8 + Q^{10})$$
$$+ 2A_3A_4(400R^2 - 800 - 100R^6 + 400R^6)\,(2Q^4 - 7Q^5 + 8Q^6 - 2Q^7 - 2Q^8 + Q^9)$$

$$
\begin{aligned}
(w^{IIQ})^2 \\
&= A_1^2(R^2-4R^4+2R^5+4R^6-4R^7+R^8)\,(4-48Q+192Q^2-288Q^3+144Q^4) \\
&+ A_2^2(R^2-4R^4+2R^5+4R^6-4R^7+R^8)\,(16-144Q+324Q^2160Q^3-740Q^4 \\
&+ 400Q^6) \\
&+ A_3^2(5.44R^2-15.556R^4+15.778R^6-6.667R^8+R^{10})\,(4-48Q+192Q^2-288Q^3 \\
&+ 144Q^4) \\
&+ A_4^2(5.44R^2-15.556R^4+15.778R^6-6.667R^8+R^{10})\,(16-144Q+324Q^2160Q^3 \\
&- 740Q^4+400Q^6)
\end{aligned}
$$

$$
\begin{aligned}
+2A_1A_2&(R^2-4R^4+2R^5+4R^6-4R^7+R^8)\,(8-84Q+264Q^2-176Q^3-240Q^4 \\
&+ 240Q^5) \\
&+ 2A_1A_3(2.33R^2-8R^4+2.33R^5-7.667R^6-3.333R^7-2R^8+R^9)\,(4-48Q+192Q^2 \\
&- 288Q^3+144Q^4) \\
&+ 2A_1A_4(2.33R^2-8R^4+2.33R^5-7.667R^6-3.333R^7-2R^8+R^9)\,(8-84Q+264Q^2 \\
&- 176Q^3-240Q^4+240Q^5)
\end{aligned}
$$

$$
\begin{aligned}
+2A_2A_3&(2.33R^2-8R^4+2.33R^5-7.667R^6-3.333R^7-2R^8+R^9)\,(8-84Q+264Q^2 \\
&- 176Q^3-240Q^4+240Q^5) \\
&+ 2A_2A_4(2.33R^2-8R^4+2.33R^5-7.667R^6-3.333R^7-2R^8 \\
&+ R^9)(16-144Q+324Q^2160Q^3-740Q^4+400Q^6) \\
&+ 2A_3A_4(5.44R^2-15.556R^4+15.778R^6-6.667R^8+R^{10})\,(8-84Q+264Q^2 \\
&- 176Q^3-240Q^4+240Q^5)
\end{aligned}
$$

$$
\begin{aligned}
(w^{IIRq})^2 \\
&= A_1^2(1-12R^2+8R^3+36R^4-48R^5+16R^6)\,(4Q^2-24Q^3+52Q^4-48Q^5+16Q^6) \\
&+ A_2^2(1-12R^2+8R^3+36R^4-48R^5+16R^6)\,(16Q^2-72Q^3+81Q^4+40Q^5-90Q^6 \\
&+ 25Q^8) \\
&+ A_3^2(5.44-46.667R^2+123.33R^4-100R^6+25R^8)\,(4Q^2-24Q^3+52Q^4-48Q^5 \\
&+ 16Q^6) \\
&+ A_4^2(5.44-46.667R^2+123.33R^4-100R^6+25R^8)\,(16Q^2-72Q^3+81Q^4+40Q^5 \\
&- 90Q^6+25Q^8)
\end{aligned}
$$

$$
\begin{aligned}
+2A_1A_2&(1-12R^2+8R^3+36R^4-48R^5+16R^6)\,(8Q^2-42Q^2+70Q^4-26Q^5-30Q^6 \\
&+ 20Q^7) \\
&+ 2A_1A_3(2.33-24R^2+9.33R^3+65R^4-40R^5-30R^6+20R^7)\,(4Q^2-24Q^3+52Q^4 \\
&- 48Q^5+16Q^6) \\
&+ 2A_1A_4(2.33-24R^2+9.33R^3+65R^4-40R^5-30R^6+20R^7)\,(8Q^2-42Q^2+70Q^4 \\
&- 26Q^5-30Q^6+20Q^7)
\end{aligned}
$$

$$+2A_2A_3(2.33-24R^2+9.33R^3+65R^4-40R^5-30R^6+20R^7)(16Q^2-72Q^3+81Q^4+40Q^5-90Q^6+25Q^8)$$
$$+2A_2A_4(2.33-24R^2+9.33R^3+65R^4-40R^5-30R^6+20R^7)(4Q^4-12Q^5+9Q^6+4Q^7-6Q^8+Q^{10})$$
$$+2A_3A_4(5.44-46.667R^2+123.33R^4-100R^6+25R^8)(8Q^2-42Q^2+70Q^4-26Q^5-30Q^6+20Q^7)$$

Integrating these five squared partial derivative equations partially with respect to R and Q in a closed domain respectively gave:

$$\int_0^1\int_0^1 (w^{IR})^2\,\partial R\partial Q = A_1^2(0.485714)(0.001587)+A_2^2(0.485714)(0.009957)+A_3^2(3.0414)(0.001587)+A_4^2(3.0414)(0.009957)+2A_1A_2(0.485714)(0.003968)+2A_1A_3(1.21428)(0.001587)+2A_1A_4(1.21428)(0.003968)+2A_2A_3(1.21428)(0.003968)+2A_2A_4(1.21428)(0.009957)+2A_3A_4(3.0414)(0.003968)$$

$$-0.000771A_1^2+0.004836A_2^2+0.004836A_3^2+0.03028A_4^2+0.003855A_1A_2+0.003854A_1A_3+0.009637A_1A_4+0.009637A_2A_3+0.024181A_2A_4+0.024137A_3A_4$$

$$\int_0^1\int_0^1 (w^{IIR})^2\,\partial R\partial Q = A_1^2(4.8)(0.001587)+A_2^2(4.8)(0.009957)+A_3^2(30.47619)(0.001587)+A_4^2(30.47619)(0.009957)+2A_1A_2(4.8)(0.003968)+2A_1A_3(12)(0.001587)+2A_1A_4(12)(0.003968)+2A_2A_3(12)(0.003968)+2A_2A_4(12)(0.009957)+2A_3A_4(30.47619)(0.003968)$$

$$=0.007618A_1^2+0.047794A_2^2+0.048366A_3^2+0.303451A_4^2+0.038093A_1A_2+0.038088A_1A_3+0.095232A_1A_4+0.095232A_2A_3+0.238968A_2A_4+0.241859A_3A_4$$

$$\int_0^1\int_0^1 (w^{IIQ})^2\partial R\partial Q = A_1^2(0.049206)(0.8) + A_2^2(0.049206)(5.142857) + A_3^2(0.30784)(0.8) + A_4^2(0.30784)(5.142857) + 2A_1A_2(0.049206)(2) + 2A_1A_3(0.123016)(0.8) + 2A_1A_4(0.123016)(2) + 2A_2A_3(0.123016)(2) + 2A_2A_4(0.123016)(5.142857) + 2A_3A_4(0.30784)(2)$$

$$= 0.039365A_1^2 + 0.253059A_2^2 + 0.246272A_3^2 + 1.583177A_4^2 + 0.196824A_1A_2 + 0.196826A_1A_3 + 0.492064A_1A_4 + 0.492064A_2A_3 + 1.265307A_2A_4 + 1.23136A_3A_4$$

$$\int_0^1\int_0^1 (w^{IIRQ})^2\partial R\partial Q = A_1^2(0.485714)(0.019048) + A_2^2(0.485714)(0.120635) + A_3^2(3.047619)(0.019048) + A_4^2(3.047619)(0.120635) + 2A_1A_2(0.485714)(0.047619) + 2A_1A_3(1.214286)(0.019048) + 2A_1A_4(1.214286)(0.047619) + 2A_2A_3(1.214286)(0.047619) + 2A_2A_4(1.214286)(0.120635) + 2A_3A_4(3.047619)(0.047619)$$

$$= 0.009252A_1^2 + 0.058594A_2^2 + 0.058051A_3^2 + 0.36765A_4^2 + 0.046258A_1A_2 + 0.046259A_1A_3 + 0.115646A_1A_4 + 0.1156464A_2A_3 + 0.292971A_2A_4 + 0.290249A_3A_4$$

Substituting the above integrals into equations (99a) gave

$$\prod_x = \frac{D}{2b^2}\Big(\frac{1}{p^3}[0.007618A_1^2 + 0.047794A_2^2 + 0.048366A_3^2 + 0.303451A_4^2 + 0.038093A_1A_2 + 0.038088A_1A_3 + 0.095232A_1A_4 + 0.095232A_2A_3 + 0.238968A_2A_4 + 0.241859A_3A_4] + p[0.039365A_1^2 + 0.253059A_2^2 + 0.246272A_3^2 + 1.583177A_4^2 + 0.196824A_1A_2 + 0.196826A_1A_3 + 0.492064A_1A_4 + 0.492064A_2A_3 + 1.265307A_2A_4 + 1.23136A_3A_4] + \frac{2}{p}[0.009252A_1^2 + 0.058594A_2^2 + 0.058051A_3^2 + 0.36765A_4^2 + 0.046258A_1A_2 + 0.046259A_1A_3 + 0.115646A_1A_4 + 0.1156464A_2A_3 + 0.292971A_2A_4 + 0.290249A_3A_4]\Big) - \frac{N_x}{2p}(0.000771A_1^2 + 0.004836A_2^2 + 0.004836A_3^2 + 0.03028A_4^2 + 0.003855A_1A_2 + 0.003854A_1A_3 + 0.009637A_1A_4 + 0.009637A_2A_3 + 0.024181A_2A_4 + 0.024137A_3A_4) \qquad (123b)$$

3.6 TOTAL POTENTIAL ENERGY FUNCTIONAL FOR CSSS PLATE

For M = N = 4

Differentiating equation(111) partially with respect to either R or Q both gave the following equations

$$w^{IR} = A(1 - 6R^2 + 4R^3)(1.5Q^2 - 2.5Q^3 + Q^4)$$

$$w^{IIR} = A(12R^2 - 12R)(1.5Q^2 - 2.5Q^3 + Q^4)$$

$$w^{IQ} = A(R - 2R^3 + R^4)(3Q - 7.5Q^2 + 4Q^3)$$

$$w^{IIQ} = A(R - 2R^3 + R^4)(3 - 15Q + 12Q^2)$$

$$w^{IIRQ} = A(1 - 6R^2 + 4R^3)(3Q - 7.5Q^2 + 4Q^3)$$

Squaring each of the above equations gave the following equations respectively:

$$(w^{IR})^2 = A^2(1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)(2.25Q^4 - 7.5Q^5 + 9.25Q^6 - 5Q^7 + Q^8)$$

$$(w^{IIR})^2 = A^2(144R^4 - 288R^3 + 144R^2)(2.25Q^4 - 7.5Q^5 + 9.25Q^6 - 5Q^7 + Q^8)$$

$$(w^{IQ})^2 = A^2(R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)(9Q^2 - 45Q^3 + 80.25Q^4 - 60Q^5 + 16Q^6)$$

$$(w^{IIQ})^2 = A^2(R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)(9 - 90Q + 297Q^2 - 360Q^3 + 144Q^4)$$

$$(w^{IIRQ})^2 = A^2(1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)(9Q^2 - 45Q^3 + 80.25Q^4 - 60Q^5 + 16Q^6)$$

Integrating these five squared partial differential equations partially with respect to R and Q in a closed domain respectively gave:

$$\int_0^1 \int_0^1 (w^{IR})^2 \partial R \partial Q = A^2(0.48571)(0.00754) = 0.0036623A^2$$

$$\int_0^1 \int_0^1 (w^{IIR})^2 \partial R \partial Q = A^2(4.8)(0.00754) = 0.036192A^2$$

$$\int_0^1 \int_0^1 (w^{IQ})^2 \partial R \partial Q = A^2(0.049206)(0.08571) = 0.00421745A^2$$

$$\int_0^1 \int_0^1 (w^{IIQ})^2 \partial R \partial Q = A^2(0.049206)(1.8) = 0.088571A^2$$

$$\int_0^1 \int_0^1 (w^{IIRQ})^2 \partial R \partial Q = A^2(0.48571)(0.085714) = 0.0416321A^2$$

Substituting the above integrals into equations (99) and (100) gave

$$\prod_x = \frac{DA^2}{2Pa^2}(0.036192 + 0.088571P^4 + 0.0832643P^2) - \frac{N_xA^2}{2P}(0.0036623) \quad (125)$$

$$\prod_y = \frac{DPA^2}{2b^2}\left(\frac{0.036192}{P^4} + 0.088571 + \frac{0.0832643}{P^2}\right) - \frac{N_yPA^2}{2}(0.00421745) \quad (126)$$

3.7 TOTAL POTENTIAL ENERGY FUNCTIONAL FOR CCSC

For M = N = 4

Differentiating equation(112) partially with respect to either R or Q or both gave the following equations:

$$w^{IR} = A(2R - 6R^2 + 4R^3)(1.5R^2 - 2.5R^3 + R^4)$$

$$w^{IIR} = A(2 - 12R + 12R^2)(1.5R^2 - 2.5R^3 + R^4)$$

$$w^{IQ} = A(R^2 - 2R^3 + R^4)(3R - 7.5R^2 + 4R^3)$$

$$w^{IIQ} = A(R^2 - 2R^3 + R^4)(3 - 15R + 12R^2)$$

$$w^{IIRQ} = A(2R - 6R^2 + 4R^3)(3R - 7.5R^2 + 4R^3)$$

Squaring each of the above equations gave the following equations respectively

$$(w^{IR})^2 = A^2(4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6)(2.25Q^4 - 7.5Q^5 + 9.25Q^6 - 5Q^7 + Q^8)$$

$$(w^{IIR})^2 = A^2(4 - 48R + 192R^2 - 288R^3 + 144R^2)(2.25Q^4 - 7.5Q^5 + 9.25Q^6 - 5Q^7 + Q^8)$$

$$(w^{IQ})^2 = A^2(R^4 - 4R^5 + 6R^6 - 4R^7 + R^8)(9Q^2 - 45Q^3 + 80.25Q^4 - 60Q^5 + 16Q^6)$$

$$(w^{IIQ})^2 = A^2(R^4 - 4R^5 + 6R^6 - 4R^7 + R^8)(9 - 90Q + 297Q^2 - 360Q^3 + 144Q^4)$$

$$(W^{IIRQ})^2 = A^2(4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6)(9Q^2 - 45Q^3 + 80.25Q^4 - 60Q^5 + 16Q^6)$$

Integrating these five squared equations partially with respect to R and Q in a closed domain respectively gave:

$$\int_0^1 \int_0^1 (w^{IR})^2 \partial R \partial Q = A^2(0.01905)(0.00754) = 0.00014367A^2$$

$$\int_0^1 \int_0^1 (w^{IIR})^2 \partial R \partial Q = A^2(0.8)(0.00754) = 0.006032A^2$$

$$\int_0^1 \int_0^1 (w^{IQ})^2 \partial R \partial Q = A^2(0.001587)(0.08571) = 0.000136022A^2$$

$$\int_0^1 \int_0^1 (w^{IIQ})^2 \partial R \partial Q = A^2(0.001587)(3.285714) = 0.005214428A^2$$

$$\int_0^1 \int_0^1 (w^{IIRQ})^2 \partial R \partial Q = A^2(0.019048)(0.085714) = 0.00163268A^2$$

Substituting the above integrals into equations (99) and (100) gave

$$\Pi_x = \frac{DA^2}{2Pa^2}(0.006032 + 0.005214428P^4 + 0.00326536P^2) - \frac{N_x A^2}{2P}(0.00014367) \qquad (127)$$

$$\Pi_y = \frac{DPA^2}{2b^2}\left(\frac{0.006032}{P^4} + 0.005214428 + \frac{0.00326536}{P^2}\right) - \frac{N_y PA^2}{2}(0.000136022) \qquad (128)$$

3.8 TOTAL POTENTIAL FUNCTIONAL FOR CCSS PLATE

For M = N = 4

Differentiating equations (113) partially with respect to either R or Q both gave the following equations

$$w^{IR} = A(3R - 7.5R^2 + 4R^3)(1.5Q^2 - 2.5Q^3 + Q^4)$$

$$w^{IIR} = A(3 - 15R + 12R^2)(1.5Q^2 - 2.5Q^3 + Q^4)$$

$$w^{IQ} = A(1.5R^2 - 2.5R^3 + R^4)(3Q - 7.5Q^2 + 4Q^3)$$

$$w^{IIQ} - A(1.5R^2 - 2.5R^3 + R^4)(3 - 15Q - 12Q^2)$$

$$w^{IIRQ} = A(3R - 7.5R^2 + 4R^3)(3Q - 7.5Q^2 + 4Q^3)$$

Squaring each of the above equations gave the following equations respectively:

$$(w^{IR})^2 = A^2(9R^2 - 45R^3 + 80.25R^4 - 60R^5 + 16R^6)(2.25Q^4 - 7.5Q^5 + 9.25Q^6 - 5Q^7 + Q^8)$$

$$(w^{IIR})^2 = A^2(9 - 90R + 297R^2 - 360R^3 + 144R^2)(2.25Q^4 - 7.5Q^5 + 9.25Q^6 - 5Q^7 + Q^8)$$

$$(w^{IQ})^2 = A^2(2.25R^4 - 7.5R^5 + 9.25R^6 - 5R^7 + R^8)(9Q^2 - 45Q^3 + 80.25Q^4 - 60Q^5 + 16Q^6)$$

$$(w^{IIQ})^2 = A^2(2.25R^4 - 7.5R^2 + 9.25R^6 - 5R^7 + R^8)(9 - 90Q + 297Q^2 - 360Q^3 + 144Q^4)$$

$$(W^{IIRQ})^2 = A^2(9R^2 - 45R^3 + 80.25R^4 - 60R^5 + 16R^6)(9Q^2 - 45Q^3 + 80.25Q^4 - 60Q^5 + 16Q^6)$$

Integrating these five squared partial differential equations partially with respect to R and Q in a closed domain respectively gave:

$$\int_0^1 \int_0^1 (w^{IR})^2 \partial R \partial Q = A^2(0.085714)(0.00754) = 0.0006462836A^2$$

$$\int_0^1 \int_0^1 (w^{IIR})^2 \partial R \partial Q - A^2(1.8)(0.00754) - 0.013572A^2$$

$$\int_0^1 \int_0^1 (w^{IQ})^2 \partial R \partial Q = A^2(0.00754)(0.085714) = 0.0006462836A^2$$

$$\int_0^1 \int_0^1 (w^{IIQ})^2 \partial R \partial Q = A^2(0.00754)(1.8) = 0.013572A^2$$

$$\int_0^1 \int_0^1 (w^{IIRQ})^2 \partial R \partial Q = A^2(0.085714)(0.085714) = 0.0073469A^2$$

Substituting the above integrals in equations (99) and (100) gave

$$\Pi_x = \frac{DA^2}{2Pa^2}(0.013572 + 0.013572P^4 + 0.01469P^2) - \frac{N_x A^2}{2P}(0.0006462836) \quad (129a)$$

$$\Pi_y = \frac{DPA^2}{2b^2}\left(\frac{0.013572}{P^4} + 0.013572 + \frac{0.01469}{P^2}\right) - \frac{N_y PA^2}{2}(0.0006462836) \quad (130)$$

For M = N = 5

Differentiating equation(113a) partially with respect to either R or Q or both gave the following equations

$$\begin{aligned} w^{IR} &= A_1(3R - 7.5R^2 + 4R^3)(1.5Q^2 - 2.5Q^3 + Q^4) \\ &+ A_2(3R - 7.5R^2 + 4R^3)(3.5Q^2 - 4.5Q^3 + Q^5) \\ &+ A_3(7R - 13.5R^2 + 5R^4)(1.5Q^2 - 2.5Q^3 + Q^4) \\ &+ A_4(7R - 13.5R^2 + 5R^4)(3.5Q^2 - 4.5Q^3 + Q^5) \end{aligned}$$

$$\begin{aligned} w^{IIR} &= A_1(3 - 15R + 12R^2)(1.5Q^2 - 2.5Q^3 + Q^4) \\ &+ A_2(3 - 15R + 12R^2)(3.5Q^2 - 4.5Q^3 + Q^5) \\ &+ A_3(7 - 27R + 20R^3)(1.5Q^2 - 2.5Q^3 + Q^4) \\ &+ A_4(7 - 27R + 20R^3)(3.5Q^2 - 4.5Q^3 + Q^5) \end{aligned}$$

$$\begin{aligned} w^{IIQ} &= A_1(1.5R^2 - 2.5R^3 + R^4)(3 - 15Q + 12Q^2) \\ &+ A_2(1.5R^2 - 2.5R^3 + R^4)(3 - 15Q + 12Q^2) \\ &+ A_3(3.5R^2 - 4.5R^3 + R^5)(2 - 12Q + 12Q^2) \\ &+ A_4(3.5R^2 - 4.5R^3 + R^5)(7 - 27Q + 20Q^3) \end{aligned}$$

$$
\begin{aligned}
w^{IIRQ} \\
&= A_1(3R - 7.5R^2 + 4R^3)(3Q - 7.5Q^2 + 4Q^3) \\
&+ A_2(3R - 7.5R^2 + 4R^3)(7Q - 13.5Q^2 + 5Q^4) \\
&+ A_3(7R - 13.5R^2 + 5R^4)(3Q - 7.5Q^2 + 4Q^3) \\
&+ A_4(7R - 13.5R^2 + 5R^4)(7Q - 13.5Q^2 + 5Q^4)
\end{aligned}
$$

Squaring each of the above five equations gave the equations respectively

$$
\begin{aligned}
(w^{IR})^2 \\
&= A_1^2(9R^2 - 45R^3 + 80.25R^4 - 60R^5 + 16R^6)(2.25Q^4 - 7.5Q^5 + 9.25Q^6 - 5Q^7 + Q^8) \\
&+ A_2^2(9R^2 - 45R^3 + 80.25R^4 - 60R^5 + 16R^6)(12.25Q^4 - 13.5Q^5 + 20.25Q^6 + 7Q^7 \\
&- 9Q^8 + Q^{10}) \\
&+ A_3^2(49R^2 - 189R^3 + 182.25R^4 - 70R^5 - 135R^6 + 25R^8)(2.25Q^4 - 7.5Q^5 + 9.25Q^6 \\
&- 5Q^7 + Q^8) \\
&+ A_4^2(49R^2 - 189R^3 + 182.25R^4 - 70R^5 - 135R^6 + 25R^8)(12.25Q^4 - 13.5Q^5 \\
&+ 20.25Q^6 + 7Q^7 - 9Q^8 + Q^{10})
\end{aligned}
$$

$$
\begin{aligned}
&+2A_1A_2(9R^2 - 45R^3 + 80.25R^4 - 60R^5 + 16R^6)(5.25Q^4 - 15.5Q^5 + 14.75Q^6 - 3Q^7 \\
&- 2.5Q^8 + Q^9) \\
&+ 2A_1A_3(21R^2 - 93R^3 + 129.25R^4 - 39R^5 - 37.5R^6 + 20R^7)(2.25Q^4 - 7.5Q^5 \\
&+ 9.25Q^6 - 5Q^7 + Q^8) \\
&+ 2A_1A_4(21R^2 - 93R^3 + 129.25R^4 - 39R^5 - 37.5R^6 + 20R^7)(5.25Q^4 - 15.5Q^5 \\
&+ 14.75Q^6 - 3Q^7 - 2.5Q^8 + Q^9)
\end{aligned}
$$

$$
\begin{aligned}
&+2A_2A_3(21R^2 - 93R^3 + 129.25R^4 - 39R^5 - 37.5R^6 + 20R^7)(5.25Q^4 - 15.5Q^5 \\
&+ 14.75Q^6 - 3Q^7 - 2.5Q^8 + Q^9) \\
&+ 2A_2A_4(21R^2 - 93R^3 + 129.25R^4 - 39R^5 - 37.5R^5 + 20R^7)(12.25Q^4 - 13.5Q^5 \\
&+ 20.25Q^6 + 7Q^7 - 9Q^8 + Q^{10}) \\
&+ 2A_3A_4(49R^2 - 189R^3 + 182.25R^4 - 70R^5 - 135R^6 + 25R^8)(5.25Q^4 - 15.5Q^5 \\
&+ 14.75Q^6 - 3Q^7 - 2.5Q^8 + Q^9)
\end{aligned}
$$

$$
\begin{aligned}
(w^{IIR})^2 \\
&= A_1^2(9 - 90R + 297R^2 - 360R^3 + 144R^4)(2.25Q^4 - 7.5Q^5 + 9.25Q^6 - 5Q^7 + Q^8) \\
&+ A_2^2(9 - 90R + 297R^2 - 360R^3 + 144R^4)(12.25Q^4 - 13.5Q^5 + 20.25Q^6 + 7Q^7 - 9Q^8 \\
&+ Q^{10}) \\
&+ A_3^2(49 - 378R + 792R^2 + 280R^3 - 1080R^4 + 400R^6)(2.25Q^4 - 7.5Q^5 + 9.25Q^6 \\
&- 5Q^7 + Q^8) \\
&+ A_4^2(49 - 378R + 792R^2 + 280R^3 - 1080R^4 + 400R^6)(12.25Q^4 - 13.5Q^5 + 20.25Q^6 \\
&+ 7Q^7 - 9Q^8 + Q^{10})
\end{aligned}
$$

$$
\begin{aligned}
&+2A_1A_2(9-\ 90R+297R^2-360R^3+144R^4)\,(5.25Q^4-15.5Q^5+14.75Q^6-3Q^7\\
&\quad -2.5Q^8+Q^9)\\
&\quad +2A_1A_3(21-\ 186R+549R^2-324R^3-300R^4+240R^5)\ \ (2.25Q^4-7.5Q^5+9.25Q^6\\
&\quad -5Q^7+Q^8)\\
&\quad +2A_1A_4(21-\ 186R+549R^2-324R^3-300R^4+240R^5)\ \ (5.25Q^4-15.5Q^5\\
&\quad +14.75Q^6-3Q^7-2.5Q^8+Q^9)
\end{aligned}
$$

$$
\begin{aligned}
&+2A_2A_3(21-\ 186R+549R^2-324R^3-300R^4+240R^5)\,(5.25Q^4-15.5Q^5+14.75Q^6\\
&\quad -3Q^7-2.5Q^8+Q^9)\\
&\quad +2A_2A_4(21-\ 186R+549R^2-324R^3-300R^4+240R^5)\,(12.25Q^4-13.5Q^5\\
&\quad +20.25Q^6+7Q^7-9Q^8+Q^{10})\\
&\quad +2A_3A_4(49-\ 378R+792R^2+280R^3-1080R^4+400R^6)\,(5.25Q^4-15.5Q^5\\
&\quad +14.75Q^6-3Q^7-2.5Q^8+Q^9)
\end{aligned}
$$

$$
\begin{aligned}
&(w^{IIRQ})^2\\
&\quad =A_1^2(9R^2-\ 45R^3+80.25R^4-60R^5+16R^6)\,(9Q^2-\ 45Q^3+80.25Q^4-60Q^5\\
&\quad +16Q^6)\\
&\quad +A_2^2(9R^2-\ 45R^3+80.25R^4-60R^5+16R^6)\,(49Q^2-\ 189Q+182.25Q^4-70Q^5\\
&\quad -135Q^6+25Q^8)\\
&\quad +A_3^2(49R^2-\ 189R^3+182.25R^4-70R^5-135R^6+25R^8)\,(9Q^2-\ 45Q^3+80.25Q^4\\
&\quad -60Q^5+16Q^6)\\
&\quad +A_4^2(49R^2-\ 189R^3+182.25R^4-70R^5-135R^6+25R^8)\,(49Q^2-\ 189Q+182.25Q^4\\
&\quad -70Q^5-135Q^6+25Q^8)
\end{aligned}
$$

$$
\begin{aligned}
&+2A_1A_2(9R^2-\ 45R^3+80.25R^4-60R^5+16R^6)\,(21Q^2-\ 93Q^3+129.25Q^4-39Q^5\\
&\quad -37.5Q^6+20Q^7)\\
&\quad +2A_1A_3(21R^2-\ 93R^3+129.25R^4-39R^5-37.5R^6+20R^7)\ \ (9Q^2-\ 45Q^3+80.25Q^4\\
&\quad -60Q^5+16Q^6)\\
&\quad +2A_1A_4(21R^2-\ 93R^3+129.25R^4-39R^5-37.5R^6+20R^7)\,(21Q^2-\ 93Q^3\\
&\quad +129.25Q^4-39Q^5-37.5Q^6+20Q^7)
\end{aligned}
$$

$$
\begin{aligned}
&+2A_2A_3(21R^2-\ 93R^3+129.25R^4-39R^5-37.5R^6+20R^7)\,(21Q^2-\ 93Q^3+129.25Q^4\\
&\quad -39Q^5-37.5Q^6+20Q^7)\\
&\quad +2A_2A_4(21R^2-93R^3+129.25R^4-39R^5-37.5R^5+20R^7)\,(49Q^2-\ 189Q\\
&\quad +182.25Q^4-70Q^5-135Q^6+25Q^8)\\
&\quad +2A_3A_4(49R^2-189R^3+182.25R^4-70R^5-135R^6+25R^8)(21Q^2-93Q^3\\
&\quad +129.25Q^4-39Q^5-37.5Q^6+20Q^7)
\end{aligned}
$$

Integrating these three squared partial derivative equations partially with respect to R and Q in a closed domain respectively gave:

$$\int_0^1 \int_0^1 (w^{IR})^2 \partial R \partial Q = A_1^2(0.085714)(0.00754) + A_2^2(0.085714)(0.058766) + A_3^2(0.692063)(0.00754) + A_4^2(0.692063)(0.058766) + 2A_1A_2(0.085714)(0.021032) + 2A_1A_3(0.242857)(0.00754) + 2A_1A_4(0.242857)(0.021032) + 2A_2A_3(0.242857)(0.021032) + 2A_2A_4(0.242857)(0.058766) + 2A_3A_4(0.692063)(0.021032)$$

$$= 0.000646A_1^2 + 0.005037A_2^2 + 0.005218A_3^2 + 0.04067A_4^2 + 2A_1A_2(0.001803) + 2A_1A_3(0.001831) + 2A_1A_4(0.005108) + 2A_2A_3(0.005108) + 2A_2A_4(0.014272) + 2A_3A_4(0.014555)$$

$$\int_0^1 \int_0^1 (w^{IIR})^2 \partial R \partial Q = A_1^2(1.8)(0.00754) + A_2^2(1.8)(0.058766) + A_3^2(14.142857)(0.00754) + A_4^2(14.142857)(0.058766) + 2A_1A_2(1.8)(0.021032) + 2A_1A_3(10)(0.00754) + 2A_1A_4(10)(0.021032) + 2A_2A_3(10)(0.021032) + 2A_2A_4(10)(0.058766) + 2A_3A_4(14.142857)(0.021032)$$

$$= 0.013572A_1^2 + 0.122143A_2^2 + 0.106637A_3^2 + 0.959692A_4^2 + 2A_1A_2(0.037858) + 2A_1A_3(0.0754) + 2A_1A_4(0.21032) + 2A_2A_3(0.21032) + 2A_2A_4(0.67857) + 2A_3A_4(0.297453)$$

$$\int_0^1\int_0^1 (w^{IIQ})^2 \partial R \partial Q = A_1^2(1.8)(0.00754) + A_2^2(1.8)(0.058766)$$
$$+ A_3^2(14.142857)(0.00754) + A_4^2(14.142857)(0.058766)$$
$$+ 2A_1A_2(1.8)(0.021032) + 2A_1A_3(10)(0.00754) + 2A_1A_4(10)(0.021032)$$
$$+ 2A_2A_3(10)(0.021032) + 2A_2A_4(10)(0.058766)$$
$$+ 2A_3A_4(14.142857)(0.021032)$$

$$= 0.013572A_1^2 + 0.122143A_2^2 + 0.106637A_3^2 + 0.959692A_4^2$$
$$+ 2A_1A_2(0.037858) + 2A_1A_3(0.0754) + 2A_1A_4(0.21032)$$
$$+ 2A_2A_3(0.21032) + 2A_2A_4(0.67857) + 2A_3A_4(0.297453)$$

$$\int_0^1\int_0^1 (w^{IIRQ})^2 \partial R \partial Q = A_1^2(0.085714)(0.085714) + A_2^2(0.085714)(0.692063)$$
$$+ A_3^2(0.692063)(0.085714) + A_4^2(0.692063)(0.692063)$$
$$+ 2A_1A_2(0.085714)(0.242857) + 2A_1A_3(0.242857)(0.085714)$$
$$+ 2A_1A_4(0.242857)(0.242857) + 2A_2A_3(0.242857)(0.242857)$$
$$+ 2A_2A_4(0.242857)(0.692063) + 2A_3A_4(0.692063)(0.692063)$$

$$= 0.007347A_1^2 + 0.059319A_2^2 + 0.059319A_3^2 + 0.478951A_4^2 + 0.041632A_1A_2 + 0.041632A_1A_3 + 0.117959A_1A_4 + 0.117959A_2A_3 + 0.336145A_2A_4 + 0.336145A_3A_4$$

$$\prod_x = \frac{D}{2b^2}\Big(\frac{1}{p^3}[0.013572A_1^2 + 0.122143A_2^2 + 0.106637A_3^2 + 0.959692A_4^2 + 2A_1A_2(0.037858) + 2A_1A_3(0.0754) + 2A_1A_4(0.21032) + 2A_2A_3(0.21032) + 2A_2A_4(0.67857) + 2A_3A_4(0.297453)] + p[0.013572A_1^2 + 0.122143A_2^2 + 0.106637A_3^2 + 0.959692A_4^2 + 2A_1A_2(0.037858) + 2A_1A_3(0.0754) + 2A_1A_4(0.21032) + 2A_2A_3(0.21032) + 2A_2A_4(0.67857) + 2A_3A_4(0.297453)] + \frac{2}{p}[0.007347A_1^2 + 0.059319A_2^2 + 0.059319A_3^2 + 0.478951A_4^2 + 0.041632A_1A_2 + 0.041632A_1A_3 + 0.117959A_1A_4 + 0.117959A_2A_3 + 0.336145A_2A_4 + 0.336145A_3A_4]\Big) - \frac{N_x}{2p}(0.000646A_1^2 + 0.005037A_2^2 + 0.005218A_3^2 + 0.04067A_4^2 + 2A_1A_2(0.001803) + 2A_1A_3(0.001831) + 2A_1A_4(0.005108) + 2A_2A_3(0.005108) + 2A_2A_4(0.014272) + 2A_3A_4(0.014555)) \qquad (129b)$$

3.9 TOTAL POTENTIAL FUNCTIONAL FOR CSFS PLATE

For M = N = 4

Differentiating equation(114) partially with respect to either R or Q both gave the following equations:

$$w^{IR} = (1 - 6R^2 + 4R^3)(C_2Q^2 + C_3Q^3 + C_4Q^4)$$

$$w^{IIR} = (12R^2 - 12R)(C_2Q^2 + C_3Q^3 + C_4Q^4)$$

$$w^{IQ} = (R - 2R^3 + R^4)(2C_2Q + 3C_3Q^2 + 4C_4Q^3)$$

$$w^{IIQ} = (R - 2R^3 + R^4)\,(2C_2 + 6C_3Q + 12C_4Q^2)$$

$$w^{IIRQ} = (1 - 6R^2 + 4R^3)\,(2C_2Q + 3C_3Q^2 + 4C_4Q^3)$$

Squaring each of the above equations gave the following equations respectively

$$(w^{IR})^2 = (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)\,(C_2^2Q^4 + 2C_2C_3Q^5 + 2C_2C_4Q^6 + C_3^2Q^6 + 2C_3C_4Q^7 + C_4^2Q^8)$$

$$(w^{IIR})^2 = (144R^4 - 288R^3 + 144R^2)(C_2^2Q^4 + 2C_2C_3Q^5 + 2C_2C_4Q^6 + C_3^2Q^6 + 2C_3C_4Q^7 + C_4^2Q^8)$$

$$(w^{IQ})^2 = (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)\,(4C_2^2Q^2 + 12C_2C_3Q^3 + 16C_2C_4Q^4 + 9C_3^2Q^4 + 24C_3C_4Q^5 + 16C_4^2Q^6)$$

$$(w^{IIQ})^2 = (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)(4C_2^2 + 24C_2C_3Q + 48C_2C_4Q^2 + 36C_3^2Q^2 + 144C_3C_4Q^3 + 144C_4^2Q^4)$$

$$(W^{IIRQ})^2 = (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)(4C_2^2Q^2 + 12C_2C_3Q^3 + 16C_2C_4Q^4 + 9C_3^2\square^4 + 24C_3C_4Q^5 + 16C_4^2Q^6)$$

Integrating these five squared partial differential equations partially with respect to R and Q in a closed domain respectively gave:

$$\int_0^1\int_0^1 (w^{IR})^2\,\partial R\partial Q = 0.485714(0.2C_2^2 + 0.14285714C_3^2 + 0.11111C_4^2 + 0.333C_2C_3 + 0.285714C_2C_4 + 0.25C_3C_4)$$

$$= 0.09714C_2^2 + 0.06939C_3^2 + 0.05397C_4^2 + 0.1619C_2C_3 + 0.13878C_2C_4 + 0.12143C_3C_4$$

$\int_0^1 \int_0^1 (w^{IIR})^2 \partial R \partial Q = 4.8\,(0.2C_2^2 + 0.14285714C_3^2 + 0.11111C_4^2 + 0.333C_2C_3 + 0.285714C_2C_4 + 0.25C_3C_4$
$)$

$$= 0.96C_2^2 + 0.68571C_3^2 + 0.5333C_4^2 + 1.6C_2C_3 + 1.37143C_2C_4 + 1.2C_3C_4$$

$$\int_0^1 \int_0^1 (w^{IQ})^2 \partial R \partial Q$$

$$= 0.04921(1.333C_2^2 + 1.8C_3^2 + 2.28571C_4^2 + 3C_2C_3 + 3.2C_2C_4 + 4C_3C_4)$$

$$= 0.06561C_2^2 + 0.08858C_3^2 + 0.11248C_4^2 + 0.14763C_2C_3 + 0.15747C_2C_4 + 0.19684C_3C_4$$

$$\int_0^1 \int_0^1 (w^{IIQ})^2 \partial R \partial Q$$

$$= 0.04921(4C_2^2 + 12C_3^2 + 28.8C_4^2 + 12C_2C_3 + 16C_2C_4 + 36C_3C_4)$$

$$= 0.19684C_2^2 + 0.59052C_3^2 + 1.41725C_4^2 + 0.59052C_2C_3 + 0.78736C_2C_4 + 1.77156C_3C_4$$

$$\int_0^1 \int_0^1 (w^{IIRQ})^2 \partial R \partial Q =$$

$$0.485714(1.333C_2^2 + 1.8C_3^2 + 2.28571C_4^2 + 3C_2C_3 + 3.2C_2C_4 + 4C_3C_4)$$

$$= 0.64762C_2^2 + 0.87429C_3^2 + 1.1102C_4^2 + 1.45714C_2C_3 + 1.55428C_2C_4 + 1.94286C_3C_4$$

Substituting the above integrals into equations (99) and (100) gave

$$\Pi_x = \frac{D}{2Pa^2}\left[(0.96 + 0.19684P^4 + 1.29524P^2)C_2^2 + (0.68571 + 0.59052P^4 + 1.74858P^2)C_3^2 + (0.53333 + 1.41725P^4 + 2.2204P^2)C_4^2 + (1.6 + 0.59052P^4 + 2.91428P^2)C_2C_3 + (0.285714 + 0.78736P^4 + 3.10856P^2)C_2C_4 + (0.25 + 1.77156P^4 + 3.88572P^2)C_3C_4\right] - \frac{N_x}{2P}(0.09714C_2^2 + 0.06939C_3^2 + 0.05397C_4^2 + 0.1619C_2C_3 + 0.13878C_2C_4 + 0.12143C_3C_4) \qquad (131)$$

$$\Pi_y = \frac{DP}{2b^2}\left[\left(\frac{0.96}{P^4} + 0.19684 + \frac{1.29524}{p^2}\right)C_2^2 + \left(\frac{0.68571}{p^4} + 0.59052 + \frac{1.74858}{P^2}\right)C_3^2 + \left(\frac{0.53333}{P^4} + 1.41725 + \frac{2.2204}{P^2}\right)C_4^2 + \left(\frac{1.6}{P^4} + 0.59052 + \frac{2.91428}{P^2}\right)C_2C_3 + \left(0.285714 + 0.78736 + \frac{3.10856}{P^2}\right)C_2C_4 + \left(0.25 + 1.77156 + \frac{3.88572}{P^2}\right)C_3C_4\right] - \frac{N_Y P}{2}(0.06561C_2^2 + 0.08858C_3^2 + 0.11248C_4^2 + 0.14763C_2C_3 + 0.15747C_2C_4 + 0.1968C_3C_4) \qquad (132)$$

3.10 TOTAL POTENTIAL FUNCTIONAL FOR SSFS PLATE

For M = N = 4

Differentiating equation(115) partially with respect to either R or Q both gave the following equations:

$$w^{IR} = (1 - 6R^2 + 4R^3)\,(C_1 Q + C_3 Q^3 + C_4 Q^4)$$

$$w^{IIR} = (12R^2 - 12R)\,(C_1 Q + C_3 Q^3 + C_4 Q^4)$$

$$w^{IQ} = (R - 2R^3 + R^4)\,(C_1 + 3C_3 Q^2 + 4C_4 Q^3)$$

$$w^{IIQ} = (R - 2R^3 + R^4)\,(6C_3 Q + 12C_4 Q^2)$$

$$w^{IIRQ} = (1 - 6R^2 + 4R^3)\,(C_1 + 3C_3 Q^2 + 4C_4 Q^3)$$

Squaring each of the above equations gave the following equations respectively

$$\begin{aligned}(w^{IR})^2 &\\ &= (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)\,(C_1^2 Q^2 + 2C_1 C_3 Q^4 + 2C_1 C_4 Q^5 \\ &\quad + C_3^2 Q^6 + 2C_3 C_4 Q^7 + C_4^2 Q^8)\end{aligned}$$

$$\begin{aligned}(w^{IIR})^2 &\\ &= (144R^4 - 288R^3 + 144R^2)(C_1^2 Q^2 + 2C_1 C_3 Q^4 + 2C_1 C_4 Q^5 + C_3^2 Q^6 \\ &\quad + 2C_3 C_4 Q^7 + C_4^2 Q^8)\end{aligned}$$

$$\begin{aligned}(w^{IQ})^2 &\\ &= (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)\,(C_1^2 + 6C_1 C_3 Q^2 + 8C_1 C_4 Q^3 + 9C_3^2 Q^4 \\ &\quad + 24C_3 C_4 Q^5 + 16C_4^2 Q^6)\end{aligned}$$

$$\begin{aligned}(w^{IIQ})^2 &\\ &= (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)(36C_3^2 Q^2 + 144C_3 C_4 Q^3 + 144C_4^2 Q^4)\end{aligned}$$

$$\begin{aligned}(W^{IIRQ})^2 &\\ &= (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)(C_1^2 + 6C_1 C_3 Q^2 + 8C_1 C_4 Q^3 \\ &\quad + 9C_3^2 Q^4 + 24C_3 C_4 Q^5 + 16C_4^2 Q^6)\end{aligned}$$

Integrating these five squared partial differential equations partially with respect to R and Q in a closed domain respectively gave

$$\int_0^1\int_0^1 (w^{IR})^2 \partial R \partial Q = 0.485714(0.3333C_1^2 + 0.14286C_3^2 + 0.11111C_4^2 + 0.4C_1C_3 + 0.3333C_1C_4 + 0.25C_3C_4)$$

$$= 0.1619C_1^2 + 0.06939C_3^2 + 0.05397C_4^2 + 0.19429C_1C_3 + 0.1619C_1C_4 + 0.12143C_3C_4$$

$$\int_0^1\int_0^1 (w^{IIR})^2 \partial R \partial Q = 4.8\,(0.3333C_1^2 + 0.14286C_3^2 + 0.11111C_4^2 + 0.4C_1C_3 + 0.3333C_1C_4 + 0.25C_3C_4)$$

$$= 1.6C_1^2 + 0.68573C_3^2 + 0.5333C_4^2 + 1.92C_1C_3 + 1.6C_1C_4 + 1.2C_3C_4$$

$$\int_0^1\int_0^1 (w^{IQ})^2 \partial R \partial Q = 0.04921(C_1^2 + 1.8C_3^2 + 2.28571C_4^2 + 2C_1C_3 + 2C_1C_4 + 4C_3C_4)$$
$$= 0.04921C_1^2 + 0.08858C_3^2 + 0.11248C_4^2 + 0.09842C_1C_3 + 0.09842C_1C_4 + 0.19684C_3C_4$$

$$\int_0^1\int_0^1 (w^{IIQ})^2 \partial R \partial Q = 0.04921(12C_3^2 + 36C_3C_4 + 28.8C_4^2)$$
$$= 0.59052C_3^2 + 1.77156C_3C_4 + 1.41725C_4^2$$

$$\int_0^1\int_0^1 (w^{IIRQ})^2 \partial R \partial Q = 0.485714(C_1^2 + 1.8C_3^2 + 2.28571C_4^2 + 2C_1C_3 + 2C_1C_4 + 4C_3C_4)$$
$$= 0.485714C_1^2 + 0.87429C_3^2 + 1.1102C_4^2 + 0.97143C_1C_3 + 0.97143C_1C_4 + 1.94286C_3C_4$$

Substituting the above integrals into equations (99) and (100) gave

$$\Pi_x = \frac{D}{2Pa^2}\left[(1.6+0.97143P^2)C_1^2 + (0.68573+0.59052P^4+1.74858P^2)C_3^2 + (0.5333+1.41725P^4+2.2204P^2)C_4^2 + (1.92+1.94286P^2)C_1C_3 + (1.6+1.94286P^2)C_1C_4 + (1.2+1.77156P^4+3.88572P^2)C_3C_4\right] - \frac{N_x}{2P}(0.1619C_1^2+0.06939C_3^2+0.05397C_4^2+0.19429C_1C_3+0.1619C_1C_4 + 0.12143C_3C_4) \qquad (133)$$

$$\Pi_y = \frac{DP}{2b^2}\left[\left(\frac{1.6}{P^4}+\frac{0.97143}{P^2}\right)C_1^2 + \left(\frac{0.68573}{P^4}+0.59052+\frac{1.74858}{P^2}\right)C_3^2 + \left(\frac{0.53333}{P^4}+1.41725+\frac{2.2204}{P^2}\right)C_4^2 + \left(\frac{1.92}{P^4}+\frac{1.94286}{P^2}\right)C_1C_3 + \left(\frac{1.6}{P^4}+\frac{1.94286}{P^2}\right)C_1C_4 + \left(\frac{1.2}{P^4}+1.77156+\frac{3.88572}{P^2}\right)C_3C_4\right] - \frac{N_YP}{2}(0.04921C_1^2+0.08858C_3^2+0.11248C_4^2+0.09842C_1C_3 + 0.09842C_1C_4+0.19684C_3C_4) \qquad (134)$$

3.11 TOTAL POTENTIAL FUNCTIONAL FOR CCFC PLATE

For M = N = 4

Differentiating equation(117) partially with respect to either R or Q both gave the following equations

$$w^{IR} = (2R-6R^2+4R^3)(C_2Q^2+C_3Q^3+C_4Q^4)$$

$$w^{IIR} = (2-12R+12R^2)(C_2Q^2+C_3Q^3+C_4Q^4)$$

$$w^{IQ} = (R^2-2R^3+R^4)(2C_2Q+3C_3Q^2+4C_4Q^3)$$

$$w^{IIQ} = (R^2-2R^3+R^4)(2C_2+6C_3Q+12C_4Q^2)$$

$$w^{IIRQ} = (2R - 6R^2 + 4R^3)(2C_2Q + 3C_3Q^2 + 4C_4Q^3)$$

Squaring each of the above equations gave the following equations respectively

$$(w^{IR})^2 = (4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6)(C_2^2Q^4 + 2C_2C_3Q^5 + 2C_2C_4Q^6 + C_3^2Q^6 + 2C_3C_4Q^7 + C_4^2Q^8)$$

$$(w^{IIR})^2 = (4 - 48R + 192R^2 - 288R^3 + 144R^2)(C_2^2Q^4 + 2C_2C_3Q^5 + 2C_2C_4Q^6 + C_3^2Q^6 + 2C_3C_4Q^7 + C_4^2Q^8)$$

$$(w^{IQ})^2 = (R^4 - 4R^5 + 6R^6 - 4R^7 + R^8)(4C_2^2Q^2 + 12C_2C_3Q^3 + 16C_2C_4Q^4 + 9C_3^2Q^4 + 24C_3C_4Q^5 + 16C_4^2Q^6)$$

$$(w^{IIQ})^2 = (R^4 - 4R^5 + 6R^6 - 4R^7 + R^8)(4C_2^2 + 24C_2C_3Q + 48C_2C_4Q^2 + 36C_3^2Q^2 + 144C_3C_4Q^3 + 144C_4^2Q^4)$$

$$(W^{IIRQ})^2 = (4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6)(4C_2^2Q^2 + 12C_2C_3Q^3 + 16C_2C_4Q^4 + 9C_3^2Q^4 + 24C_3C_4Q^5 + 16C_4^2Q^6)$$

Integrating these five squared partial differential equations partially with respect to R and Q in a closed domain respectively gave

$$\int_0^1 \int_0^1 (w^{IR})^2 \partial R \partial Q = 0.01905(0.2C_2^2 + 0.14285714C_3^2 + 0.11111C_4^2 + 0.333C_2C_3 + 0.285714C_2C_4 + 0.25C_3C_4)$$

$$= 0.00381C_2^2 + 0.00272C_3^2 + 0.00212C_4^2 + 0.00635C_2C_3 + 0.00544C_2C_4 + 0.00476C_3C_4$$

$$\int_0^1 \int_0^1 (w^{IIR})^2 \partial R \partial Q = 0.8(0.2C_2^2 + 0.14285714C_3^2 + 0.11111C_4^2 + 0.333C_2C_3 + 0.285714C_2C_4 + 0.25C_3C_4)$$

$$= 0.16C_2^2 + 0.11429C_3^2 + 0.08889C_4^2 + 0.26667C_2C_3 + 0.22857C_2C_4 + 0.2C_3C_4$$

$$\int_0^1 \int_0^1 (w^{IQ})^2 \partial R \partial Q = 0.001587(1.333C_2^2 + 1.8C_3^2 + 2.28571C_4^2 + 3C_2C_3 + 3.2C_2C_4 + 4C_3C_4)$$

$$= 0.00212C_2^2 + 0.00286C_3^2 + 0.00363C_4^2 + 0.00476C_2C_3 + 0.00508C_2C_4 + 0.00635C_3C_4$$

$$\int_0^1 \int_0^1 (w^{IIQ})^2 \partial R \partial Q = 0.001587(4C_2^2 + 12C_3^2 + 28.8C_4^2 + 12C_2C_3 + 16C_2C_4 + 36C_3C_4)$$

$$= 0.00635C_2^2 + 0.01904C_3^2 + 0.04571C_4^2 + 0.01904C_2C_3 + 0.02539C_2C_4 + 0.05713C_3C_4$$

$$\int_0^1 \int_0^1 (w^{IIRQ})^2 \partial R \partial Q = 0.01905(1.333C_2^2 + 1.8C_3^2 + 2.28571C_4^2 + 3C_2C_3 + 3.2C_2C_4 + 4C_3C_4)$$

$$= 0.0254C_2^2 + 0.03429C_3^2 + 0.04354C_4^2 + 0.05715C_2C_3 + 0.06096C_2C_4 + 0.0762C_3C_4$$

Substituting the above integrals into equations (99) and (100) gave

$$\Pi_x = \frac{D}{2Pa^2}[(0.16 + 0.00635P^4 + 0.0508P^2)C_2^2 + (0.11429 + 0.01904P^4 + 0.06858P^2)C_3^2 + (0.08889 + 0.04571P^4 + 0.08708P^2)C_4^2 + (0.26667 + 0.01904P^4 + 0.1143P^2)C_2C_3 + (0.22857 + 0.02539P^4 + 0.12192P^2)C_2C_4 + (0.2 + 0.05713P^4 + 0.1524P^2)C_3C_4] - \frac{N_X}{2P}(0.00381C_2^2 + 0.00272C_3^2 + 0.00212C_4^2 + 0.00635C_2C_3 + 0.00544C_2C_4 + 0.00476C_3C_4) \qquad (137)$$

$$\Pi_y = \frac{DP}{2b^2}\left[\left(\frac{0.16}{P^4} + 0.00635 + \frac{0.0508}{P^2}\right)C_2^2 + \left(\frac{0.11429}{P^4} + 0.01904 + \frac{0.06858}{P^2}\right)C_3^2 + \left(\frac{0.08889}{P^4} + 0.04571 + \frac{0.08708}{P^2}\right)C_4^2 + \left(\frac{0.26667}{P^4} + 0.01904 + \frac{0.1143}{P^2}\right)C_2C_3 + \left(0.22857 + 0.02539 + \frac{0.12192}{P^2}\right)C_2C_4 + \left(0.2 + 0.05713 + \frac{0.1524}{P^2}\right)C_3C_4\right] - \frac{N_YP}{2}(0.00212C_2^2 + 0.0286C_3^2 + 0.00363C_4^2 + 0.00476C_2C_3 + 0.00508C_2C_4 + 0.00635C_3C_4) \qquad (138)$$

3.12 MINIMIZATION OF TOTAL POTENTIAL ENERGY

Principle of variational calculus as adopted in this thesis is all about minimization of the total potential energy functional of a flat rectangular thin plate. This was achieved by differentiating the functional partially with respect to one of the coefficients (unknown constants or parameters) of the shape (displacement) function of the particular plate. The resultant partial derivative became equal to zero. Rearranging the partial derivative and making axial (in-plane) force (N_x, N_y or N_{xy} as the case may be) the subject of the equation gave what stability equation.

If there were n coefficients (parameters) in the shape function, then there must be n stability equations. This was so because, the total potential energy had to be partially differentiated, in turn, by each of the coefficients. This will result into n partial derivatives of the total potential energy functional, and each was equal to zero.

The stability equations for plates of different boundary conditions were derived under different subsections below.

3.12.1 STABILITY EQUATION FOR SSSS PLATE

Total potential energy functionals of equations (118) and (119) have one coefficient from the shape function of SSSS plate. Hence, only one partial derivative will be obtained from each case.

$$\frac{\partial \prod_x}{\partial A} = \frac{DA}{Pa^2}\left(0.23621 + 0.23621P^4 + 0.47182P^2\right) - \frac{N_x A}{P}(0.0239) = 0$$

Making N_x the subject of the equation gave

$$N_x = \frac{D}{a^2}\left(\frac{0.23621 + 0.23621P^4 + 0.47182P^2}{0.0239}\right) \qquad (139)$$

$$\frac{\partial \prod_y}{\partial A} = \frac{DPA}{b^2}\left(\frac{0.23621}{P^4} + 0.23621 + \frac{0.47182}{P^2}\right) - N_y PA(0.0239) = 0$$

Making N_y the subject of the equation gave

$$N_y = \frac{D}{b^2}\left(\frac{\left(\frac{0.23621}{P^4} + 0.23621 + \frac{0.47182}{P^2}\right)}{0.0239}\right) \tag{140}$$

Making equation(139) to be in terms of b by substituting a = P b into it gave

$$N_x = \frac{D}{b^2}\left(\frac{\frac{0.23621}{P^2} + 0.23621P^2 + 0.47182}{0.0239}\right) \tag{141}$$

3.12.2 STABILITY EQUATION FOR CCCC PLATE

Total potential energy functionals of equations (121) and (122) have one coefficient from the shape function of CCCC plate. Hence, only one partial derivative was obtained from each case.

$$\frac{\partial \prod_x}{\partial A} = \frac{DA}{Pa^2}(0.00127 + 0.00127P^4 + 0.00072P^2) - \frac{N_x A}{P}(0.00003) = 0$$

Making N_x the subject of the equation gave

$$N_x = \frac{D}{a^2}\left(\frac{0.00127 + 0.00127P^4 + 0.00073P^2}{0.00003}\right) \tag{142}$$

Substituting a = P b into equation(142) gave

$$N_x = \frac{D}{b^2}\left(\frac{\frac{0.00127}{P^2} + 0.00127P^2 + 0.00073}{0.00003}\right) \tag{143}$$

$$\frac{\partial \prod_y}{\partial A} = \frac{DPA}{b^2}\left(\frac{0.00127}{P^4} + 0.00127 + \frac{0.00073}{P^2}\right) - N_y PA(0.00003) = 0$$

Making N_y the subject of the equation gave

$$N_y = \frac{D}{b^2}\left(\frac{\left(\frac{0.00127}{P^4} + 0.00127 + \frac{0.00073}{P^2}\right)}{0.00003}\right) \tag{144}$$

3.12.3 STABILITY EQUATION FOR CSCS PLATE

For M = N = 4

Total potential energy functionals of equations (123) and (124) have one coefficient from the shape function of CSCS plate. Hence, only one partial derivative was obtained from each case.

$$\frac{\partial \prod_x}{\partial A} = \frac{DA}{Pa^2}\left(0.00763 + 0.03937P^4 + 0.0185P^2\right) - \frac{N_x A}{P}(0.00077) = 0$$

Making N_x the subject of the equation gave

$$N_x = \frac{D}{a^2}\left(\frac{0.00763 + 0.03937P^4 + 0.0185P^2}{0.00077}\right) \tag{145}$$

Substituting a = P b in this equation gave

$$N_x = \frac{D}{b^2}\left(\frac{\frac{0.00763}{P^2} + 0.03937P^2 + 0.0185}{0.00077}\right) \quad (146)$$

$$\frac{\partial \prod_y}{\partial A} = \frac{DPA}{b^2}\left(0.00763 + 0.03937 + \frac{0.0185}{P^2}\right) - N_y PA(0.00094) = 0$$

Making N_y the subject of the equation gave

$$N_y = \frac{D}{b^2}\left(\frac{\frac{0.00763}{P^4} + 0.03937 + \frac{0.0185}{P^2}}{0.00094}\right) \quad (147)$$

For M = N = 5

Total potential energy functional of equation (123b) has four coefficients from the shape function of CSCS plate, when M = N = 5. Hence, four partial derivatives were obtained from the functional when it is partially differentiated respectively with respect to each of the four coefficients (A_1, A_2, A_3 and A_4). The four partial derivatives are:

$$\begin{aligned}
&\frac{\partial \prod_x}{\partial A_1} \\
&\quad = \frac{D}{2b^2}\left(\frac{1}{p^3}[2 * 0.007618A_1 + 0.038093\,A_2 + 0.038088A_3 + 0.095232A_4]\right. \\
&\quad + p[2 * 0.039365A_1 + 0.196824A_2 + 0.196826A_3 + 0.492064A_4] \\
&\quad \left. + \frac{2}{p}[2 * 0.009252A_1 + 0.046258A_2 + 0.046259A_3 + 0.115646A_4]\right) \\
&\quad - \frac{N_x}{2p}(2 * 0.000771A_1 + 0.003855A_2 + 0.003854A_3 + 0.009637A_4) \\
&\quad = 0 \qquad (147a)
\end{aligned}$$

$$\frac{\partial \prod_x}{\partial A_2}$$
$$= \frac{D}{2b^2}\left(\frac{1}{p^3}[0.038093A_1 + 2 * 0.047794A_2 + 0.095232A_3 + 0.238968A_4]\right.$$
$$+ p[0.196824A_1 + 2 * 0.253059A_2 + 0.492064A_3 + 1.265307A_4]$$
$$\left.+ \frac{2}{p}[0.046258A_1 + 2 * 0.058594A_2 + 0.1156464A_3 + 0.292971A_4]\right)$$
$$- \frac{N_x}{2p}(0.003855A_1 + 2 * 0.004836A_2 + 0.009637A_3 + 0.024181A_4)$$
$$= 0 \qquad (147b)$$

$$\frac{\partial \prod_x}{\partial A_3}$$
$$= \frac{D}{2b^2}\left(\frac{1}{p^3}[0.038088A_1 + 0.095232A_2 + 2 * 0.048366A_3 + 0.241859A_4]\right.$$
$$+ p[0.196826A_1 + 0.492064A_2 + 2 * 0.246272A_3 + 1.23136A_4]$$
$$\left.+ \frac{2}{p}[0.046259A_1 + 0.1156464A_2 + 2 * 0.058051A_3 + 0.290249A_4]\right)$$
$$- \frac{N_x}{2p}(0.003854A_1 + 0.009637A_2 + 2 * 0.004836A_3 + 0.024137A_4)$$
$$= 0 \qquad (147c)$$

$$\frac{\partial \prod_x}{\partial A_4}$$
$$= \frac{D}{2b^2}\left(\frac{1}{p^3}[0.095232A_1 + 0.238968A_2 + 0.241859A_3 + 2 * 0.303451A_4]\right.$$
$$+ p[0.492064A_1 + 1.265307A_2 + 1.23136A_3 + 2 * 1.583177A_4]$$
$$\left.+ \frac{2}{p}[0.115646A_1 + 0.292971A_2 + 0.290249A_3 + 2 * 0.36765A_4]\right)$$
$$- \frac{N_x}{2p}(0.009637A_1 + 0.024181A_2 + 0.024137A_3 + 2 * 0.03028A_4)$$
$$= 0 \qquad (147d)$$

3.12.4 STABILITY EQUATION FOR CSSS PLATE

Total potential energy functionals of equations (125) and (126) have one coefficient from the shape function of CSSS plate. Hence, only one partial derivative was obtained from each case.

$$\frac{\partial \prod_x}{\partial A} = \frac{DA}{Pa^2}\left(0.036192 + 0.088571P^4 + 0.0832643P^2\right) - \frac{N_x A}{P}(0.0036623) = 0$$

Making N_x the subject of the equation gave

$$N_x = \frac{D}{a^2}\left(\frac{0.036192 + 0.088571P^4 + 0.0832643P^2}{0.0036623}\right) \qquad (148)$$

Substituting a = P b in this equation gave

$$N_x = \frac{D}{b^2}\left(\frac{\frac{0.036192}{P^2} + 0.088571P^2 + 0.0832643}{0.0036623}\right) \qquad (149)$$

$$\frac{\partial \prod_y}{\partial A} = \frac{DAP}{b^2}\left(\frac{0.036192}{P^4} + 0.088571 + \frac{0.0832643}{P^2}\right) - N_y PA(0.00421745) = 0$$

Making N_y the subject of the equation gave

$$N_y = \frac{D}{b^2}\left(\frac{\frac{0.036192}{P^4} + 0.088571 + \frac{0.0832643}{P^2}}{0.00421745}\right) \qquad (150)$$

3.12.5 STABILITY EQUATION FOR CCSC PLATE

Total potential energy functional of equations (127) and (128) have one coefficient from the shape function of ccsc plate. Hence, only one partial derivative was obtained from each case.

$$\frac{\partial \prod_x}{\partial A} = \frac{DA}{Pa^2}(0.006032 + 0.005214428P^4 + 0.00326536P^2) - \frac{N_x A}{P}(0.00014367) = 0$$

Making N_x the subject of the equation gave

$$N_x = \frac{D}{a^2}\left(\frac{0.006032 + 0.005214428P^4 + 0.00326536P^2}{0.00014367}\right) \qquad (151)$$

Substituting a = P b into equation(151) gave

$$N_x = \frac{D}{b^2}\left(\frac{\frac{0.006032}{P^2} + 0.005214428P^2 + 0.00326536}{0.00014367}\right) \qquad (152)$$

$$\frac{\partial \prod_y}{\partial A} = \frac{DPA}{b^2}\left(\frac{0.006032}{P^4} + 0.005214428 + \frac{0.00326536}{P^2}\right) - N_y PA(0.000136022) = 0$$

Making N_y the subject of the equation gave

$$N_y = \frac{D}{b^2}\left(\frac{\frac{0.006032}{P^4} + 0.005214428 + \frac{0.00326536}{P^2}}{0.000136022}\right) \qquad (153)$$

3.12.6 STABILITY EQUATION FOR CCSS PLATE

For M = N = 4

Total potential energy functionals of equations (129) and (130) have one coefficient from the shape function of CCSS plate. Hence, only one partial derivative was obtained from each case.

$$\frac{\partial \prod_x}{\partial A} = \frac{DA}{Pa^2}\left(0.013572 + 0.013572P^4 + 0.01469P^2\right) - \frac{N_x A}{P}(0.0006462836) = 0$$

Making N_x the subject of the equation gave

$$N_x = \frac{D}{a^2}\left(\frac{0.013572 + 0.013572P^4 + 0.01469P^2}{0.0006462836}\right) \qquad (154)$$

Substituting a = P b into equation(154) gave

$$N_x = \frac{D}{b^2}\left(\frac{\frac{0.013572}{P^2} + 0.013572P^2 + 0.01469}{0.0006462836}\right) \qquad (155)$$

$$\frac{\partial \prod_y}{\partial A} = \frac{DPA}{b^2}\left(\frac{0.013572}{P^4} + 0.013572 + \frac{0.01469}{P^2}\right) - N_y PA(0.000642836) = 0$$

Making N_y the subject of equation gave

$$N_y = \frac{D}{b^2}\left(\frac{\frac{0.013572}{P^4} + 0.013572 + \frac{0.01469}{P^2}}{0.000642836}\right) \qquad (156)$$

For M = N = 5

Total potential energy functional of equation (129b) has four coefficients from the shape function of CSCS plate, when M = N = 5. Hence, four partial derivatives were obtained from the functional when it is partially differentiated respectively with respect to each of the four coefficients (A_1, A_2, A_3 and A_4). The four partial derivatives are:

$$\begin{aligned}
\frac{\partial \prod_x}{\partial A_1}
&= \frac{D}{2b^2}\left(\frac{1}{p^3}[2 * 0.013572A_1 + 2A_2(0.037858) + 2A_3(0.0754)\right.\\
&+ 2A_4(0.21032)]\\
&+ p[2 * 0.013572A_1 + A_2(0.037858) + A_3(0.0754) + A_4(0.21032)]\\
&\left.+ \frac{2}{p}[2 * 0.007347A_1 + 0.041632\,A_2 + 0.041632A_3 + 0.117959A_4]\right)\\
&- \frac{N_x}{2p}(2 * 0.000646A_1 + 2A_2(0.001803) + 2A_3(0.001831)\\
&+ 2A_4(0.005108)) = 0 \qquad (156a)
\end{aligned}$$

$$\frac{\partial \prod_x}{\partial A_2}$$
$$= \frac{D}{2b^2}\left(\frac{1}{p^3}[2A_1(0.037858) + 2 * 0.122143A_2 + 2A_3(0.21032)\right.$$
$$+ 2A_4(0.67857)]$$
$$+ p[2A_1(0.037858) + 2 * 0.122143A_2 + 2A_3(0.21032) + 2A_4(0.67857)]$$
$$\left. + \frac{2}{p}[0.041632A_1 + 2 * 0.059319A_2 + 0.117959A_3 + 0.336145A_4]\right)$$
$$- \frac{N_x}{2p}(2A_1(0.001803) + 2 * 0.005037A_2 + 2A_3(0.005108)$$
$$+ 2A_4(0.014272)) = 0 \qquad (156b)$$

$$\frac{\partial \prod_x}{\partial A_3}$$
$$= \frac{D}{2b^2}\left(\frac{1}{p^3}[2A_1(0.0754) + 2A_2(0.21032) + 2 * 0.106637A_3\right.$$
$$+ 2A_4(0.297453)]$$
$$+ p[2A_1(0.0754) + 2A_2(0.21032) + 2 * 0.106637A_3 + 2A_4(0.297453)]$$
$$\left. + \frac{2}{p}[0.041632A_1 + 0.117959A_2 + 2 * 0.059319A_3 + 0.336145A_4]\right)$$
$$- \frac{N_x}{2p}(2A_1(0.001831) + 2A_2(0.005108) + 2 * 0.005218A_3$$
$$+ 2A_4(0.014555)) = 0 \qquad (156c)$$

$$\frac{\partial \prod_x}{\partial A_4}$$
$$= \frac{D}{2b^2}\left(\frac{1}{p^3}[2A_1(0.21032) + 2A_2(0.67857) + 2A_3(0.297453) + 2\right.$$
$$* 0.959692A_4]$$
$$+ p[2A_1(0.21032) + 2A_2(0.67857) + 2A_3(0.297453) + 2 * 0.959692A_4]$$
$$\left. + \frac{2}{p}[0.117959A_1 + 0.336145A_2 + 0.336145A_3 + 2 * 0.478951A_4]\right)$$
$$- \frac{N_x}{2p}(2A_1(0.005108) + 2A_2(0.014272) + 2A_3(0.014555) + 2$$
$$* 0.04067A_4)$$
$$= 0 \qquad (156d)$$

3.12.7 STABILITY EQUATION FOR CSFS PLATE

Total potential energy functionals of equations (131) and (132) have one coefficient from the shape function of CSFC plate. Hence, only one partial derivative was obtained from each case.

$$\frac{\partial \prod_x}{\partial C_2} = \frac{D}{2Pa^2}\left[(1.92 + 0.39368P^4 + 2.59048P^2)C_2 + (1.6 + 0.59052P^4 + 2.91428P^2)C_3 + (0.285714 + 0.78736P^4 + 3.10856P^2)C_4\right] - \frac{N_x}{2P}(0.19428C_2 + 0.1619C_3 + 0.13878C_4) = 0$$

$$\frac{\partial \prod_x}{\partial C_3} = \frac{D}{2Pa^2}\left[(1.6 + 0.59052P^4 + 2.91428P^2)C_2 + (1.37142 + 1.18104P^4 + 3.49716P^2)C_3 + (0.25 + 1.77156P^4 + 3.88572P^2)C_4\right] - \frac{N_x}{2P}(0.1619C_2 + 0.13878C_3 + 0.12143C_4) = 0$$

$$\frac{\partial \prod_x}{\partial C_4} = \frac{D}{2Pa^2}\left[(0.285714 + 0.78736P^4 + 3.10856P^2)C_2 + (0.25 + 1.77156P^4 + 3.88572P^2)C_3 + (1.06667 + 2.8345P^4 + 4.4408P^2)C_4\right] - \frac{N_x}{2P}(0.13878C_2 + 0.12143C_3 + 0.10794C_4) = 0$$

These three equations can be put in matrix form as shown in equation(157)

3.12.8 STABILITY EQUATION FOR SSFS PLATE

Total potential energy functionals of equations (133) and (134) have one coefficient from the shape function of SSFS plate. Hence, only one partial derivative was obtained from each case.

$$\frac{\partial \Pi_x}{\partial C_1} = \frac{D}{2Pa^2}[(3.2+1.94286P^2)C_1+(1.92+1.94286P^2)C_3 + (1.6+1.94286P^2)C_4] - \frac{N_x}{2P}(0.3238C_1+0.19429C_3+0.1619C_4)=0$$

$$\frac{\partial \Pi_x}{\partial C_3} = \frac{D}{2Pa^2}[(1.92+1.94286P^2)C_1+(1.37146+1.1181P^4+3.49716P^2)C_3 + (1.2+1.77156P^4+3.88572P^2)C_4] - \frac{N_x}{2P}(0.19429C_1+0.13878C_3+0.12143C_4)=0$$

$$\frac{\partial \Pi_x}{\partial C_4} = \frac{D}{2Pa^2}[(1.6+1.94286P^2)C_1+(1.2+1.77156P^4+3.88572P^2)C_3 + (1.0667+2.8345P^4+4.4408P^2)C_4] - \frac{N_x}{2P}(0.1619C_1+0.12143C_3+0.10794C_4)=0$$

$$\frac{\partial \Pi_y}{\partial C_1} = \frac{DP}{2b^2}\left[\left(\frac{3.2}{P^4}+\frac{1.94286}{P^2}\right)C_1+\left(\frac{1.92}{P^4}+\frac{1.94286}{P^2}\right)C_3+\left(\frac{1.6}{P^4}+\frac{1.94286}{P^2}\right)C_4\right] - \frac{N_y}{2P}(0.09842C_1+0.09842C_3+0.09842C_4)=0$$

$$\frac{\partial \prod_y}{\partial C_3} = \frac{DP}{2b^2}\left[\left(\frac{1.92}{P^4}+\frac{1.94286}{P^2}\right)C_1+\left(\frac{1.37146}{p^4}+1.1181+\frac{3.49716}{P^2}\right)C_3 + \left(\frac{1.2}{P^4}+1.77156+\frac{3.88572}{P^2}\right)C_4\right]-\frac{N_y}{2P}(0.09842C_1+0.17716C_3 + 0.19684C_4)=0$$

$$\frac{\partial \prod_y}{\partial C_4} = \frac{DP}{2b^2}\left[\left(\frac{1.6}{P^4}+\frac{1.94286}{P^2}\right)C_1+\left(\frac{1.2}{P^4}+1.77156+\frac{3.88572}{P^2}\right)C_3 + \left(\frac{1.06667}{P^4}+2.8345+\frac{4.4408}{P^2}\right)C_4\right] - \frac{N_y}{2P}(0.09842C_1+0.19684C_3+0.22496C_4)-0$$

These equations can be put in matrix forms as shown in equation(159)

3.12.9 STABILITY EQUATION FOR CCFC PLATE

Total potential energy functionals of equations (137) and (138) have one coefficient from the shape function of CCFC plate. Hence, only one partial derivative was obtained from each case.

$$\frac{\partial \prod_x}{\partial C_2} = \frac{D}{2Pa^2}[(0.32+0.0127P^4+0.1016P^2)C_2 + (0.26667+0.01904P^4+0.1143P^2)C_3 + (0.22857+0.02539P^4+0.12192P^2)C_4] - \frac{N_x}{2P}(0.00762C_2+0.00635C_3+0.00544C_4)=0$$

$$\frac{\partial \prod_x}{\partial C_3} = \frac{D}{2Pa^2}\left[(0.26667 + 0.01904P^4 + 0.1143P^2)C_2 + (0.22858 + 0.03808P^4 + 0.13716P^2)C_3 + (0.2 + 0.05713P^4 + 0.1524P^2)C_4\right] - \frac{N_x}{2P}(0.00635C_2 + 0.00544C_3 + 0.00476C_4) = 0$$

$$\frac{\partial \prod_x}{\partial C_4} = \frac{D}{2Pa^2}\left[(0.22857 + 0.02539P^4 + 0.12192P^2)C_2 + (0.2 + 0.05713P^4 + 0.1524P^2)C_3 + (0.17778 + 0.09142P^4 + 0.17416P^2)C_4\right] - \frac{N_x}{2P}(0.00544C_2 + 0.00476C_3 + 0.00424C_4) = 0$$

$$\frac{\partial \prod_y}{\partial C_2} = \frac{DP}{2b^2}\left[\left(\frac{0.32}{P^4} + 0.0127 + \frac{0.1016}{P^2}\right)C_2 + \left(\frac{0.26667}{P^4} + 0.01904 + \frac{0.1143}{P^2}\right)C_3 + \left(\frac{0.22857}{P^4} + 0.02539 + \frac{0.12192}{P^2}\right)C_4\right] - \frac{N_y P}{2}(0.00424C_2 + 0.00476C_3 + 0.00508C_4) = 0$$

$$\frac{\partial \prod_y}{\partial C_3} - \frac{DP}{2b^2}\left[\left(\frac{0.26667}{P^4} + 0.01904 + \frac{0.1143}{P^2}\right)C_2 + \left(\frac{0.22858}{P^4} + 0.03808 + \frac{0.13716}{P^2}\right)C_3 + \left(\frac{0.2}{p^4} + 0.05713 + \frac{0.1524}{P^2}\right)C_4\right] - \frac{N_y P}{2}(0.00476C_2 + 0.00572C_3 + 0.00635C_4) - 0$$

$$\frac{\partial \prod_y}{\partial C_4} = \frac{DP}{2b^2}\left[\left(\frac{0.22857}{P^4} + 0.02539 + \frac{0.12192}{P^2}\right)C_2 + \left(\frac{0.2}{P^4} + 0.05713 + \frac{0.1524}{P^2}\right)C_3 + \left(\frac{0.17778}{P^4} + 0.09142 + \frac{0.17416}{P^2}\right)C_4\right] - \frac{N_y P}{2}(0.00508C_2 + 0.00635C_3 + 0.00726C_4) = 0$$

These equations can be put in matrix forms as shown in equation(163)

$$\frac{D}{a^2}\begin{bmatrix} (1.92+0.39368P^4+2.59048P^2) & (1.6+0.59052P^4+2.91428P^2) & (0.285714+0.78736P^4+3.10856P^2) \\ (1.6+0.59052P^4+2.91428P^2) & (1.37142+1.18104P^4+3.49716P^2) & (0.25+1.77156P^4+3.88572P^2\ ) \\ (0.285714+0.78736P^4+3.10856P^2) & (0.25+1.77156P^4+3.88572P^2\ ) & (1.06667+2.8345P^4+4.4408P^2) \end{bmatrix}$$
$$-N_x\begin{bmatrix} 0.19428 & 0.1619 & 0.13878 \\ 0.1619 & 0.13878 & 0.12143 \\ 0.13878 & 0.12143 & 0.10794 \end{bmatrix}=0 \qquad (157)$$

In the same manner, three partial derivatives from equation(132) in terms C_2, C_3 and C_4 can be put in matrix form as

$$\frac{D}{b^2}\begin{bmatrix} (1.92+0.39368P^4+2.59048P^2) & (1.6+0.59052P^4+2.91428P^2) & (0.285714+0.78736P^4+3.10856P^2) \\ (1.6+0.59052P^4+2.91428P^2) & (1.37142+1.18104P^4+3.49716P^2) & (0.25+1.77156P^4+3.88572P^2\ ) \\ (0.285714+0.78736P^4+3.10856P^2) & (0.25+1.77156P^4+3.88572P^2\ ) & (1.06667+2.8345P^4+4.4408P^2) \end{bmatrix}-$$
$$N_y\begin{bmatrix} 0.13122 & 0.14763 & 0.15747 \\ 0.14763 & 0.17716 & 0.19684 \\ 0.15747 & 0.19684 & 0.22496 \end{bmatrix}=0 \qquad (158$$

)

$$\frac{D}{a^2}\begin{bmatrix} (3.2+1.94286P^2) & (1.92+1.94286P^2) & (1.6+1.94286P^2) \\ (1.92+1.94286P^2) & (1.37146+1.1181P^4+3.49716P^2) & (1.2+1.77156P^4+3.88572P^2\ ) \\ (1.6+1.94286P^2) & (1.2+1.77156P^4+3.88572P^2\ ) & (1.0667+2.8345P^4+4.4408P^2) \end{bmatrix}$$
$$-N_x\begin{bmatrix} 0.3238 & 0.19429 & 0.1619 \\ 0.19429 & 0.13878 & 0.12143 \\ 0.1619 & 0.12143 & 0.10794 \end{bmatrix}=0 \qquad (159)$$

$$\frac{D}{b^2}\begin{bmatrix} (3.2+1.94286P^2) & (1.92+1.94286P^2) & (1.6+1.94286P^2) \\ (1.92+1.94286P^2) & (1.37146+1.1181P^4+3.49716P^2) & (1.2+1.77156P^4+3.88572P^2) \\ (1.6+1.94286P^2) & (1.2+1.77156P^4+3.88572P^2) & (1.0667+2.8345P^4+4.4408P^2) \end{bmatrix} - N_y\begin{bmatrix} 0.09842 & 0.09842 & 0.09842 \\ 0.09842 & 0.17716 & 0.19684 \\ 0.09842 & 0.19684 & 0.22496 \end{bmatrix} = 0 \qquad (160)$$

$$\frac{D}{a^2}\begin{bmatrix} (0.32+0.0127P^4+0.1016P^2) & (0.26667+0.01904P^4+0.1143P^2) & (0.22857+0.02539P^4+0.12192P^2) \\ (0.26667+0.01904P^4+0.1143P^2) & (0.22858+0.03808P^4+0.13716P^2) & (0.2+0.05713P^4+0.1524P^2) \\ (0.22857+0.02539P^4+0.12192P^2) & (0.2+0.05713P^4+0.1524P^2) & (0.17778+0.01942P^4+0.17416P^2) \end{bmatrix} - N_x\begin{bmatrix} 0.0762 & 0.00635 & 0.00544 \\ 0.00635 & 0.00544 & 0.00476 \\ 0.00544 & 0.00476 & 0.00424 \end{bmatrix} = 0 \qquad (163)$$

$$\frac{D}{b^2}\begin{bmatrix} (0.32/P^4+0.0127+0.1016/P^2) & (0.26667/P^4+0.01904+0.1143/P^2) & (0.22857/P^4+0.02539+0.12192/P^2) \\ (0.26667/P^4+0.01904+0.1143P^2) & (0.22858/P^4+0.03808+0.13716/P^2) & (0.2/P^4+0.05713+0.1524/P^2) \\ (0.22857/P^4+0.02539+0.12192/P^2) & (0.2/P^4+0.05713+0.1524/P^2) & (0.17778/P^4+0.01942+0.17416/P^2) \end{bmatrix} - N_x\begin{bmatrix} 0.0762 & 0.00635 & 0.00544 \\ 0.00635 & 0.00544 & 0.00476 \\ 0.00544 & 0.00476 & 0.00424 \end{bmatrix} - 0 \qquad (164)$$

3.13 MATRIX ITERATIVE-INVERSION

For both Eigen-value equation of consistent mass, trivial solutions will exist if and only if

$\{X\} = 0$. To avoid nontrivial solutions, the Eigen-value equation of consistent mass will give

$$|[K] - \omega^2 [M]| = 0 \qquad \text{-------------} \quad (165)$$

Equation (9) can simply be written as

$$|[A] - \lambda[B]| = 0 \qquad \text{---------------} (166)$$

$$\text{Let } [C] = \big[[A] - \lambda[B]\big] \qquad \text{---------------} (167)$$

$$\text{That is to say } |C| = 0 \qquad \text{---------------} (168)$$

The inverse of matrix [C] is denoted as $[C]^{-1}$. From elementary mathematics,

$$[C]^{-1} = \frac{[D]^T}{|C|} \qquad \text{---------------} (169)$$

Where [D] is the matrix of the cofactors of the elements of $|C|$. The implication of equation (169) is that the inverse of matrix C, $[C]^{-1}$ will be infinity (do not exist) as long as its determinant is equal to zero. It is easier to evaluate the inverse of a matrix using row operation than evaluating the determinant of the same matrix. Hence, instead of working to get determinant, one will go straight to get the inverse of the matrix. Remember, row operation will be used. At this point iteration becomes very important. One can start by putting the value of λ to zero, and checking if the inverse $[C]^{-1}$ exists or not. If the inverse does exit then zero is not the Eigen-value. λ will then be increased (say by 0.1), and used to test if the inverse of C matrix exists. If, again, the inverse does not exist, then λ will be increased and the process repeated until the matrix inverse ceases to exist. The value of λ that caused the matrix inverse to be infinity becomes the lowest Eigen-value. To get the next Eigen-value, starting Eigen-value will be the preceding Eigen-value plus say 0.1 (λ+0.1).

QBASIC PROGRAM FOR THE METHOD

To ease the use of this method, a QBASIC program was written. The program is user-friendly and interactive. It requires no special training to be used. The program is as shown below.

```
    CLS:REM Private Sub STARTMNU
10   PRINT "WHAT'S THE NO. OF ROWS OF THIS MATRIX ?": INPUT VROW
    VROW = VROW * 1
20   VCOLUMN = VROW
    REDIM INVM(20 * VROW, 20 * VROW), INVAM(20 * VROW, 20 * VROW)
    REDIM INVRM(20 * VROW, 20 * VROW), INVABM(20 * VROW, 20 * VROW)
    REDIM A(2 * VROW, 2 * VROW), B(2 * VROW, 2 * VROW)
    REDIM A1(2 * VROW, 2 * VROW), B1(2 * VROW, 2 * VROW)
```

```
    T = 0: OWUS = 0
2220  IF VROW = 0 THEN PRINT "IT CAN BE ZERO, PRESS ENTER TO CONTINUE"
  INPUT HHH: GOTO 10
2221  FOR X = 1 TO VROW
    FOR Y = 1 TO VROW
    PRINT "ENTER A("; X; ","; Y; ")": INPUT A1(X, Y)
    NEXT Y
    NEXT X
    FOR X = 1 TO VROW
    FOR Y = 1 TO VROW
    PRINT "ENTER B1("; X; ","; Y; ")": INPUT B1(X, Y)
    NEXT Y
    NEXT X
2223  FOR X = 1 TO VROW
    FOR Y = 1 TO VROW
    A(X, Y) = A1(X, Y)
    NEXT Y
    NEXT X
   FOR X = 1 TO VROW
```

```
FOR Y = 1 TO VROW
B(X, Y) = B1(X, Y)
NEXT Y
NEXT X
22555   FOR I = 1 TO VROW
FOR J = 1 TO VCOLUMN
INVM(I, J) = A(I, J) - T * B(I, J)
NEXT J
NEXT I
REM THE INVERSE IS CARRIED OUT HERE
FOR I = 1 TO VROW
FOR J = 1 TO 2 * VCOLUMN
INVAM(I, J) = 0
NEXT J
NEXT I
      FOR I = 1 TO VROW
FOR J = 1 TO 2 * VCOLUMN
INVAM(I, J) = INVAM(I, J) + INVM(I, J)
NEXT J
```

```
NEXT I
FOR I = 1 TO VROW
INVAM(I, I + VCOLUMN) = INVAM(I, I + VCOLUMN) + 1
NEXT I
FOR I = 1 TO VROW
FOR J = 1 TO VCOLUMN
INVABM(I, J) = INVAM(I, J)
NEXT J
NEXT I
ZZ = 1
REM THIS IS THE PLACE FOR INVERSION PROPER
FOR I = 1 TO VROW
OWUSS = INVAM(I, I)
3333    IF OWUSS > -.0001 AND OWUSS < .0001 THEN GOTO 2222
FOR J = 1 TO 2 * VCOLUMN
INVAM(I, J) = INVAM(I, J) / OWUSS
NEXT J
FOR J = 1 TO VROW
IF (J = I) THEN GOTO 77777
```

```
      OWUSS = INVAM(J, I)

      FOR K = 1 TO 2 * VCOLUMN

      INVAM(J, K) = INVAM(J, K) - OWUSS * INVAM(I, K)

      NEXT K

77777   NEXT J

      NEXT I

      T = T + .0001

      GOTO 22555

2222    ' HERE IS THE PLACE INTERCHANGE OF ROWS

      IF ZZ >= VROW THEN GOTO 1111111

      FOR W = 1 TO 2 * VCOLUMN

      INVRM(I, W) = INVAM(I, W): INVRM(I + ZZ, W) = INVAM(I + ZZ, W)

      INVAM(I, W) = INVRM(I + ZZ, W): INVAM(I + ZZ, W) = INVRM(I, W)

      NEXT W

      OWUSS = INVAM(I, I)

      IF OWUSS = 0 THEN ZZ = ZZ + 1: GOTO 6666

      ZZ = 1: GOTO 3333

6666    FOR W = 1 TO 2 * VCOLUMN

      INVRM(I, W) = INVAM(I, W): INVRM(I + ZZ, W) = INVAM(I + ZZ, W)
```

```
INVAM(I, W) = INVRM(I + ZZ, W): INVAM(I + ZZ, W) = INVAM(I, W)

NEXT W

ZZ = ZZ + 1: GOTO 2222

'this is the end of inversion

1111111 CLS : PRINT

PRINT "   LOWEST EIGEN-VALUE IS    "; USING "#######.###"; T
```

TESTING OF THE PROGRAM

The program will be used to test the following problems.

1. $\left\| \begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\|$ (Stroud, 1982)

2. $\left\| \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.2 \\ 0.1 & 0.2 & 0.3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\|$ (James, Smith and Wolford, 1977)

3. $\left\| \begin{bmatrix} 5.143 & 3.863 & 3.543 \\ 3.863 & 6.05 & 6.857 \\ 3.543 & 6.857 & 8.341 \end{bmatrix} \quad \lambda \begin{bmatrix} 0.324 & 0.194 & 0.162 \\ 0.194 & 0.139 & 0.121 \\ 0.162 & 0.121 & 0.108 \end{bmatrix} \right\|$

4. $\left\| \begin{bmatrix} 1.8844 & 4.7212 & 4.7212 \\ 4.7212 & 11.8316 & 11.80294 \\ 4.7212 & 11.80294 & 11.91948 \end{bmatrix} - \lambda \begin{bmatrix} 0.0478 & 0.1195 & 0.1195 \\ 0.1195 & 0.29904 & 0.299 \\ 0.1195 & 0.299 & 0.30 \end{bmatrix} \right\|$

5. $$\begin{vmatrix} \begin{bmatrix} 0.60953 & 0.3962 & 0.34287 & 0.30477 \\ 0.3962 & 0.4039 & 0.40964 & 0.41738 \\ 0.34287 & 0.40964 & 0.44353 & 0.4777 \\ 0.30477 & 0.41738 & 0.4777 & 0.53884 \end{bmatrix} - \lambda \begin{bmatrix} 0.0127 & 0.00762 & 0.00635 & 0.00544 \\ 0.00762 & 0.00544 & 0.00476 & 0.00423 \\ 0.00635 & 0.00476 & 0.00423 & 0.00381 \\ 0.00544 & 0.00423 & 0.00381 & 0.00346 \end{bmatrix} \end{vmatrix}$$

3.14 SUMMARY OF STABILITY EQUATIONS

The stability equations got so far were summarized and given in terms of π, D, b, P, and M. This was done by multiplying the stability equations by π^2/π^2 and replacing aspect ratio, P with the augmented aspect ratio, P/M. M is the number of failure mode (number of buckles) in the load direction.

SSSS RECTANGULAR PLATE

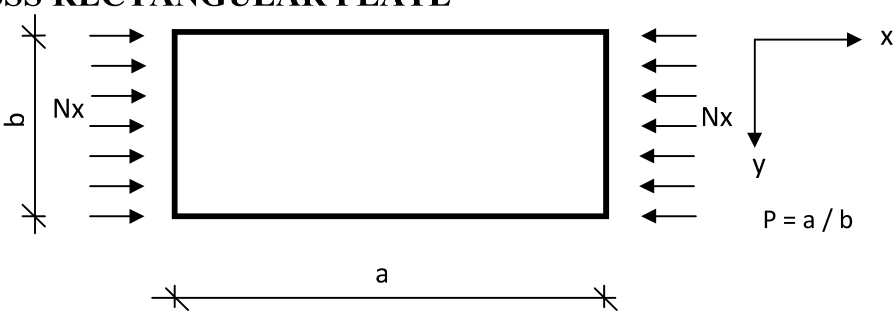

Figure 3.3: Loading arrangement for SSSS plate

For M = N = 4

Multiplying equation(141) by π^2/π^2 gave

$$N_x = \frac{D\pi^2}{b^2}\left(\frac{1.001}{P^2} + 1.001\,P^2 + 2\right) \qquad (170)$$

CCCC RECTANGULAR PLATE

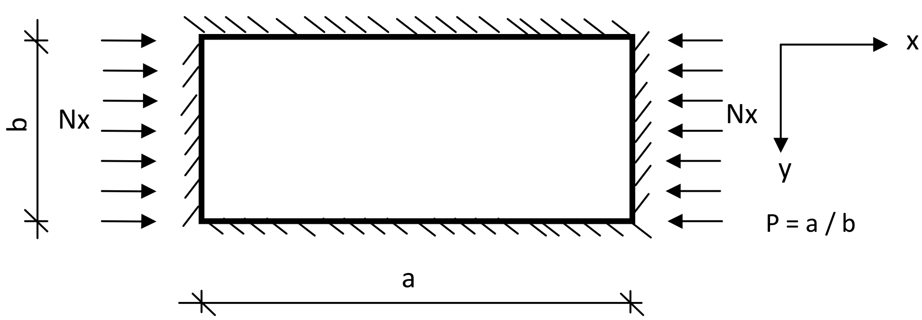

Figure 3.4: Loading arrangement for CCCC plate

For M = N = 4

Multiplying equation(143) by π^2/π^2 gave

$$N_x = \frac{D\pi^2}{b^2}\left(\frac{4.255}{P^2} + 4.255P^2 + 2.428\right) \qquad (171)$$

CSCS RECTANGULAR PLATE

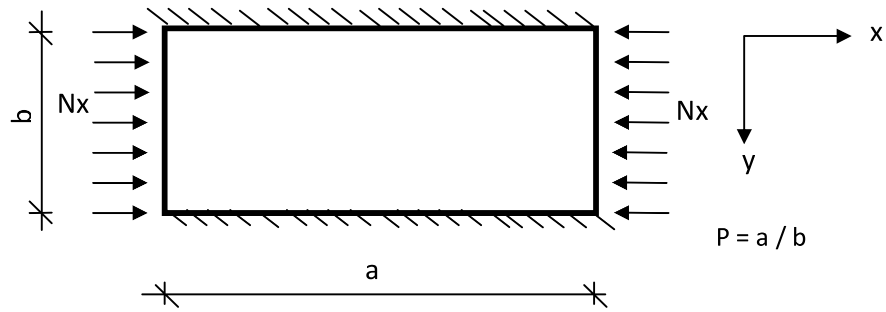

Figure 3.5: Loading arrangement for CSCS plate

For M = N = 4

Multiplying equation(146) by π^2/π^2 and gave

$$N_x = \frac{D\pi^2}{b^2}\left(\frac{1.001}{P^2} + 5.165P^2 + 2.428\right) \qquad (177)$$

SCSC RECTANGULAR PLATE

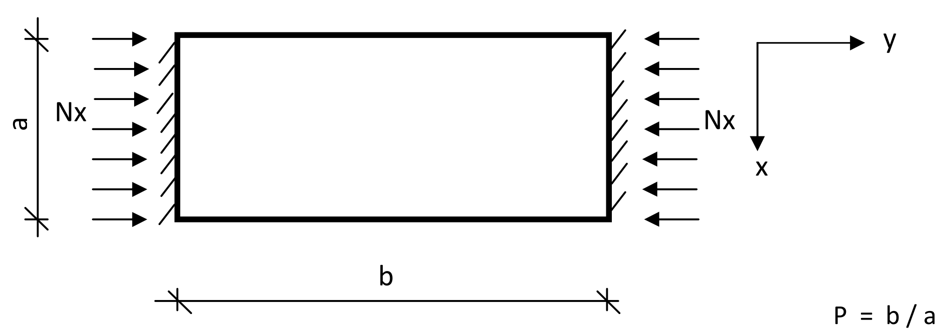

Figure 3.6: Loading arrangement for SCSC plate

For M = N = 4

Multiplying equation(147) by π^2/π^2 gave

$$N_y = \frac{D\pi^2}{b^2}\left(\frac{0.822b^4}{a^2} + 4.244 + \frac{1.994b^2}{a^2}\right)$$

Since the load was in y direction, then P = b/a and the buckling load would be in terms of a. Therefore, the above equation would be rewritten as:

$$N_y = \frac{D\pi^2}{a^2}\left(\frac{4.244}{P^2} + 0.822P^2 + 1.994\right) \qquad (183)$$

For M = N = 5

A Visual BASIC program (see PROGRAM D) was written using the four equations (147a; 147b; 147c; 147d). With this program one would calculate

K [$(N_x)cr = K\frac{\pi^2 D}{b^2}$] factor for critical buckling load of the plate when he enters the value of aspect ratio (P = a/b) in the program.

PROGRAM D (PROGRAM FOR CSCS PLATE)

```
Private Sub STARTMNU_Click()
 ReDim AA(40, 100, 100), AANS(100, 100), ANS(100, 100)
      ReDim MROW(100), MCOLUMN(100), MM(40, 100, 100), MMANS(100, 100), EMMANS(100, 100)
      ReDim INVM(100, 200), INVAM(200, 200), INVRM(200, 200), INVABM(100, 200), A(200, 200),
B(200, 200)
      Dim VROW As Variant, VCOLUMN As Variant
      Cls
      FontSize = 11: OWUS = 0
```

```
2220   OWUS = 0        ' THIS AREA IS FOR MATRIX INVERSION
       'HERE IS THE INPUT FOR INVERSION

10     VROW = 4
       If VROW = 0 Then Notice = InputBox("IT IS NOT POSSIBLE", "ROW OF MATRIX CAN'T BE ZERO",
"Click O.K. for me"): GoTo 10
20     VCOLUMN = VROW

2221   P = InputBox("WHAT IS ASPECT RATIO, P"): P = P * 1
    ' If P = 0.8 Then P = 0.8001
     'If P = 0.4 Then P = 0.4001
30
       A(1, 1) = 0.00762 + 0.03937 * P ^ 4 + 0.0185 * P ^ 2
   A(1, 2) = 0.01905 + 0.09841 * P ^ 4 + 0.04626 * P ^ 2
   A(1, 3) = 0.01904 + 0.09841 * P ^ 4 + 0.04626 * P ^ 2
   A(1, 4) = 0.04762 + 0.24603 * P ^ 4 + 0.11565 * P ^ 2
   A(2, 1) = A(1, 2)
   A(2, 2) = 0.04779 + 0.25306 * P ^ 4 + 0.11719 * P ^ 2
   A(2, 3) = 0.04762 + 0.24603 * P ^ 4 + 0.11565 * P ^ 2
   A(2, 4) = 0.11948 + 0.63265 * P ^ 4 + 0.29297 * P ^ 2
   A(3, 1) = A(1, 3)
   A(3, 2) = A(2, 3)
   A(3, 3) = 0.04837 + 0.24627 * P ^ 4 + 0.1161 * P ^ 2
   A(3, 4) = 0.1209 + 0.61568 * P ^ 4 + 0.29025 * P ^ 2
   A(4, 1) = A(1, 4)
   A(4, 2) = A(2, 4)
   A(4, 3) = A(3, 4)
   A(4, 4) = 0.303451 + 1.583177 * P ^ 4 + 0.7353 * P ^ 2

   B(1, 1) = 0.00077: B(1, 2) = 0.00193: B(1, 3) = 0.00193: B(1, 4) = 0.00482: ' B(1, 4) = 0.00482
   B(2, 1) = 0.00193: B(2, 2) = 0.00484: B(2, 3) = 0.00482: B(2, 4) = 0.0121: 'B(2, 4) = 0.01209
   B(3, 1) = 0.00193: B(3, 2) = 0.00482: B(3, 3) = 0.00483: B(3, 4) = 0.0121: 'B(3, 4) = 0.01207:'B(3, 3) = 0.00484
   B(4, 1) = 0.00482: B(4, 2) = 0.01209: B(4, 3) = 0.01207: B(4, 4) = 0.03028: 'B(4, 4) = 0.03034:'B(4, 3) = 0.01209
       Print " MATRIX A"
       For X = 1 To VROW
       For Y = 1 To VROW
       A(X, Y) = A(X, Y)
       Print A(X, Y);
       Next Y
       Print
       Next X
       Print: Print
      Print "MATRIX B"
       For X = 1 To VROW
       For Y = 1 To VROW
       B(X, Y) = B(X, Y)
       Print B(X, Y);
       Next Y
       Print
       Next X

       T = 0
22555  ' Print " T = "; T
```

```
       For I = 1 To VROW
       For J = 1 To VCOLUMN
       INVM(I, J) = A(I, J) - T * B(I, J)
       Next J
       Next I

       'THE INVERSE IS CARRIED OUT HERE
       ' THE PREAMBLE OF INVERSION

       For I = 1 To VROW
       For J = 1 To 2 * VCOLUMN
       INVAM(I, J) = 0
       Next J
       Next I

       For I = 1 To VROW
       For J = 1 To 2 * VCOLUMN
       INVAM(I, J) = INVAM(I, J) + INVM(I, J)
       Next J
       Next I

       For I = 1 To VROW
        INVAM(I, I + VCOLUMN) = INVAM(I, I + VCOLUMN) + 1
       Next I



       For I = 1 To VROW
       For J = 1 To VCOLUMN
       INVABM(I, J) = INVAM(I, J)
       Next J
       Next I

       ZZ = 1

       'THIS IS THE PLACE FOR INVERSION PROPER

       For I = 1 To VROW
       OWUSS = INVAM(I, I)
       If Abs(OWUSS) < 0.0001 Then GoTo 2222
3333   'Print " T    = "; T
       For J = 1 To 2 * VCOLUMN
       INVAM(I, J) = INVAM(I, J) / OWUSS
       Next J
       For J = 1 To VROW
       If (J = I) Then GoTo 77777
       OWUSS = INVAM(J, I)
       For K = 1 To 2 * VCOLUMN
       INVAM(J, K) = INVAM(J, K) - OWUSS * INVAM(I, K)
       Next K
77777  Next J
       Next I
```

```
      T = T + 0.01
      GoTo 22555

2222   ' HERE IS THE PLACE INTERCHANGE OF ROWS
      If I + ZZ = 3 * VROW Or I + ZZ > 3 * VROW Then GoTo 1111111: 'MsgBox (IMPOSSIBLE), , "THIS
MATRIX HAS NO INVERSE":
      For W = 1 To 2 * VCOLUMN
      INVRM(I, W) = INVAM(I, W): INVRM(I + ZZ, W) = INVAM(I + ZZ, W)
      INVAM(I, W) = INVRM(X + ZZ, W): INVAM(I + ZZ, W) = INVRM(I, W)
      Next W
      OWUSS = INVAM(I, I)
      If Abs(OWUSS) < 0.0001 Then ZZ = ZZ + 1: GoTo 6666: 'If OWUSS = 0
      ZZ = 1: GoTo 3333
6666   For W = 1 To 2 * VCOLUMN
      INVRM(I, W) = INVAM(I, W): INVRM(I + ZZ, W) = INVAM(I + ZZ, W)
      INVAM(I, W) = INVRM(I + ZZ, W): INVAM(I + ZZ, W) = INVAM(I, W)
      Next W
      If I + ZZ = 3 * VROW Or I + ZZ > 3 * VROW Then: GoTo 1111111: 'MsgBox (IMPOSSIBLE), , "THIS
MATRIX HAS NO INVERSE":
      ZZ = ZZ + 1: GoTo 2222
      'this is the end of inversion
1111111 'Cls
      Print "RESULT"
      T = T / (22 / 7) ^ 2 / P ^ 2
      Print "    P   =", P, "    H  =  "; Format(T, "0.##");
End Sub
```

CSSS RECTANGULAR PLATE

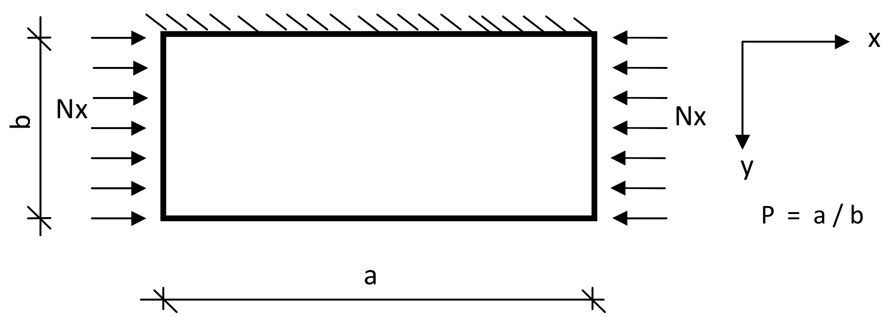

Figure 3.7: Loading arrangement for CSSS plate

For M = N = 4

Multiplying equation(149) by π2 /π2 gave

$$N_x = \frac{D\pi^2}{b^2}\left(\frac{1.001}{P^2} + 2.45P^2 + 2.304\right) \qquad (189)$$

SCSS RECTANGULAR PLATE

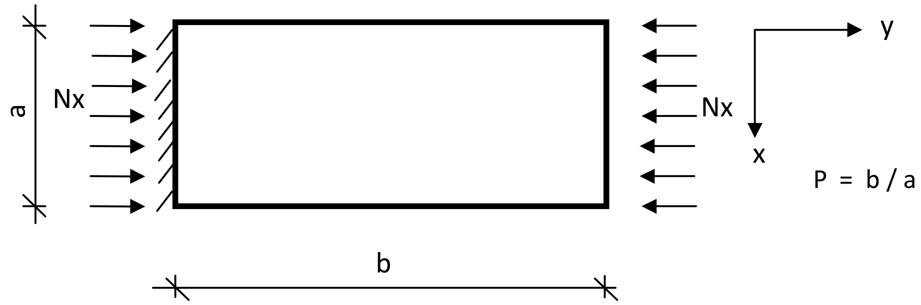

Figure 3.8: Loading arrangement for SCSS plate

For M = N = 4

Multiplying equation(150) by π^2/π^2 gave

$$N_y = \frac{D\pi^2}{b^2}\left(\frac{0.8695b^4}{a^4} + 2.1279 + \frac{2.0004b^2}{a^2}\right)$$

Since the load was in y direction, then P = b/a and the buckling load would be in terms of a. Therefore, the above equation would be rewritten as:

$$N_y = \frac{D\pi^2}{a^2}\left(\frac{2.1279}{P^2} + 0.8695P^2 + 2.0004\right) \quad (196)$$

CCSC RECTANGULAR PLATE

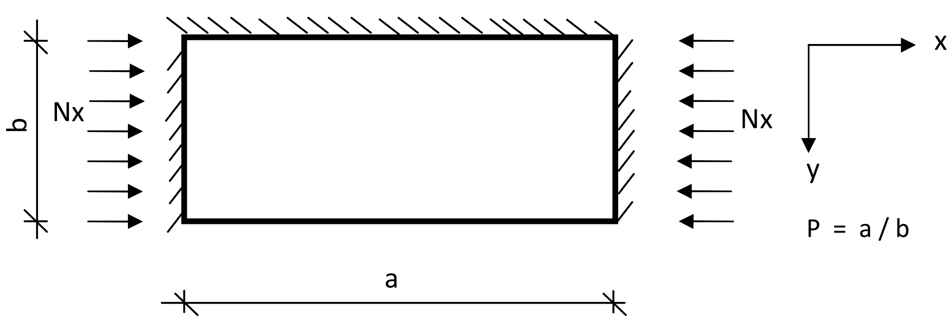

Figure 3.9: Loading arrangement for CCSC plate

For M = N = 4

Multiplying equation(152) by π2 /π2 and gave

$$N_x = \frac{D\pi^2}{b^2}\left(\frac{4.254}{P^2} + 3.6774P^2 + 2.303\right) \qquad (201)$$

CCSS RECTANGULAR PLATE

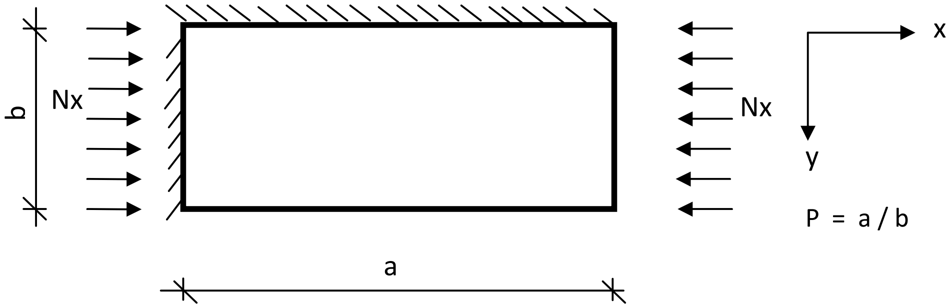

Figure 3.10: Loading arrangement for CCSS plate

For M = N = 4

Multiplying equation(155) by π2 /π2 gave

$$N_x = \frac{D\pi^2}{b^2}\left(\frac{2.1278}{P^2} + 2.1278P^2 + 2.303\right) \qquad (208)$$

For M = N = 5

A Visual BASIC program (see PROGRAM E) was written using the four equations (156a; 156b; 156c; 156d). With this program one would calculate

K [$(N_x)cr - K\frac{\pi^2 D}{b^2}$] factor for critical buckling load of the plate when he enters the value of aspect ratio (P = a/b) in the program.

PROGRAM E (PROGRAM FOR CCSS PLATE)

```
Private Sub STARTMNU_Click()
```

```
 ReDim AA(40, 100, 100), AANS(100, 100), ANS(100, 100)
       ReDim MROW(100), MCOLUMN(100), MM(40, 100, 100), MMANS(100, 100),
EMMANS(100, 100)
       ReDim INVM(100, 200), INVAM(200, 200), INVRM(200, 200), INVABM(100, 200), A(200,
200), B(200, 200)
       Dim VROW As Variant, VCOLUMN As Variant
       Cls
       FontSize = 11: OWUS = 0

2220    OWUS = 0
       ' THIS AREA IS FOR MATRIX INVERSION
       'HERE IS THE INPUT FOR INVERSION

10      VROW = 4
       If VROW = 0 Then Notice = InputBox("IT IS NOT POSSIBLE", "ROW OF MATRIX CAN'T BE
ZERO", "Click O.K. for me"): GoTo 10
20      VCOLUMN = VROW

2221    P = InputBox("WHAT IS ASPECT RATIO, P"): P = P * 1
     ' If P = 0.8 Then P = 0.8001
      'If P = 0.4 Then P = 0.4001
30
       A(1, 1) = 0.013572 + 0.013572 * P ^ 4 + 0.014694 * P ^ 2
    A(1, 2) = 0.037858 + 0.037858 * P ^ 4 + 0.041632 * P ^ 2
    A(1, 3) = 0.0754 + 0.0754 * P ^ 4 + 0.041632 * P ^ 2
    A(1, 4) = 0.21032 + 0.21032 * P ^ 4 + 0.117959 * P ^ 2
    A(2, 1) = A(1, 2)
    A(2, 2) = 0.122143 + 0.122143 * P ^ 4 + 0.118638 * P ^ 2
    A(2, 3) = 0.21032 + 0.21032 * P ^ 4 + 0.117959 * P ^ 2
    A(2, 4) = 0.67857 + 0.67857 * P ^ 4 + 0.336145 * P ^ 2
    A(3, 1) = A(1, 3)
    A(3, 2) = A(2, 3)
    A(3, 3) = 0.106637 + 0.106637 * P ^ 4 + 0.059319 * P ^ 2
    A(3, 4) = 0.297453 + 0.297453 * P ^ 4 + 0.336145 * P ^ 2
    A(4, 1) = A(1, 4)
    A(4, 2) = A(2, 4)
    A(4, 3) = A(3, 4)
    A(4, 4) = 0.959692 + 0.959692 * P ^ 4 + 0.478951 * P ^ 2

    B(1, 1) = 0.000646: B(1, 2) = 0.001803: B(1, 3) = 0.001831: B(1, 4) = 0.005108
    B(2, 1) = 0.001803: B(2, 2) = 0.005037: B(2, 3) = 0.005108: B(2, 4) = 0.014272: 'B(2, 2) =
0.00582
    B(3, 1) = 0.00183: B(3, 2) = 0.00511: B(3, 3) = 0.00522: B(3, 4) = 0.014555
    B(4, 1) = 0.005108: B(4, 2) = 0.014272: B(4, 3) = 0.014555: B(4, 4) = 0.04067
       Print " MATRIX A"
       For X = 1 To VROW
       For Y = 1 To VROW
       A(X, Y) = A(X, Y)
```

```
Print A(X, Y);
Next Y
Print
Next X
Print: Print
Print "MATRIX B"
For X = 1 To VROW
For Y = 1 To VROW
B(X, Y) = B(X, Y)
Print B(X, Y);
Next Y
Print
Next X


T = 0
22555  ' Print " T = "; T
For I = 1 To VROW
For J = 1 To VCOLUMN
INVM(I, J) = A(I, J) - T * B(I, J)
Next J
Next I

'THE INVERSE IS CARRIED OUT HERE
' THE PREAMBLE OF INVERSION

For I = 1 To VROW
For J = 1 To 2 * VCOLUMN
INVAM(I, J) = 0
Next J
Next I

For I = 1 To VROW
For J = 1 To 2 * VCOLUMN
INVAM(I, J) = INVAM(I, J) + INVM(I, J)
Next J
Next I

For I = 1 To VROW
 INVAM(I, I + VCOLUMN) = INVAM(I, I + VCOLUMN) + 1
Next I

      For I = 1 To VROW
For J = 1 To VCOLUMN
INVABM(I, J) = INVAM(I, J)
Next J
Next I
```

```
      ZZ = 1

      'THIS IS THE PLACE FOR INVERSION PROPER
       For I = 1 To VROW
      OWUSS = INVAM(I, I)
      If Abs(OWUSS) < 0.0001 Then GoTo 2222
3333    'Print " T     = "; T
      For J = 1 To 2 * VCOLUMN
      INVAM(I, J) = INVAM(I, J) / OWUSS
      Next J
      For J = 1 To VROW
      If (J = I) Then GoTo 77777
      OWUSS = INVAM(J, I)
      For K = 1 To 2 * VCOLUMN
      INVAM(J, K) = INVAM(J, K) - OWUSS * INVAM(I, K)
      Next K
77777   Next J
      Next I
      T = T + 0.01
      GoTo 22555

2222    ' HERE IS THE PLACE INTERCHANGE OF ROWS
      If I + ZZ = 3 * VROW Or I + ZZ > 3 * VROW Then GoTo 1111111: 'MsgBox (IMPOSSIBLE), ,
"THIS MATRIX HAS NO INVERSE":
      For W = 1 To 2 * VCOLUMN
      INVRM(I, W) = INVAM(I, W): INVRM(I + ZZ, W) = INVAM(I + ZZ, W)
      INVAM(I, W) = INVRM(X + ZZ, W): INVAM(I + ZZ, W) = INVRM(I, W)
      Next W
      OWUSS = INVAM(I, I)
     ' If OWUSS = 0 Then ZZ = ZZ + 1: GoTo 6666
      If Abs(OWUSS) < 0.0001 Then ZZ = ZZ + 1: GoTo 6666: 'If OWUSS = 0
      ZZ = 1: GoTo 3333
6666    For W = 1 To 2 * VCOLUMN
      INVRM(I, W) = INVAM(I, W): INVRM(I + ZZ, W) = INVAM(I + ZZ, W)
      INVAM(I, W) = INVRM(I + ZZ, W): INVAM(I + ZZ, W) = INVAM(I, W)
      Next W
      If I + ZZ = 3 * VROW Or I + ZZ > 3 * VROW Then: GoTo 1111111: 'MsgBox (IMPOSSIBLE), ,
"THIS MATRIX HAS NO INVERSE":
      ZZ = ZZ + 1: GoTo 2222
      'this is the end of inversion
1111111 'Cls
      Print "RESULT"
      T = T / (22 / 7) ^ 2 / P ^ 2
      Print "     P   =", P, "   H  =  "; Format(T, "0.##");
End Sub
```

CSFS RECTANGULAR PLATE

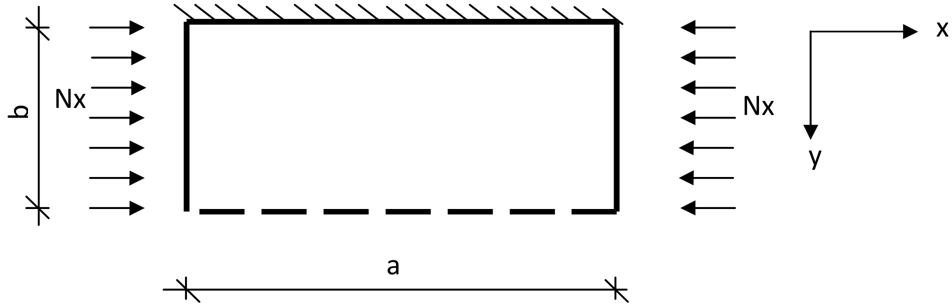

Figure 3.11: Loading arrangement for CSFS plate

For M = N = 4

Equation(157) is a 3 x 3 matrix equation that needed to be solved for the Eigen-value. BASIC program was written, as shown in PROGRAM A, to carry out the Eigen-value computation.

PROGRAM A (PROGRAM FOR CSFS PLATE)

```
        NN = 3
    REDIM AA(12, 12, 12), AANS(12, 12), ANS(12, 12)
    REDIM INVM(12, 12), INVAM(12, 12), INVRM(12, 12), INVABM(12, 12), A(12, 12), B(12, 12)
     REDIM  A1(12, 12), B1(12, 12)
    CLS
    OW = 1: T = 0
     'HERE IS THE INPUT FOR INVERSION

10     FOR P = .1 TO 1.4 STEP .1
        T = 0
    A1(1, 1) = 1.92 + .39368 * P ^ 4 + 2.59048 * P ^ 2
    A1(1, 2) = 1.6 + .59052 * P ^ 4 + 2.91428 * P ^ 2
    A1(1, 3) = .285714 + .78736 * P ^ 4 + 3.10856 * P ^ 2
    A1(2, 2) = 1.37142 + 1.18104 * P ^ 4 + 3.49716 * P ^ 2
    A1(2, 3) = .25 + 1.77156 * P ^ 4 + 3.88572 * P ^ 2
    A1(3, 3) = 1.06667 + 2.8345 * P ^ 4 + 4.4408 * P ^ 2
    A1(2, 1) = A1(1, 2): A1(3, 1) = A1(1, 3): A1(3, 2) = A(2, 3)
    B1(1, 1) = .19428: B1(1, 2) = .1619: B1(1, 3) = .13878
    B1(2, 2) = .13878: B1(2, 3) = .12143: B1(3, 3) = .10794
    B1(2, 1) = .1619: B1(3, 1) = .13878: B1(3, 2) = .12143
20    FOR X = 1 TO NN
    FOR Y = 1 TO NN
    A(X, Y) = A1(X, Y)
    NEXT Y
    NEXT X
```

```
      FOR X = 1 TO NN
      FOR Y = 1 TO NN
      B(X, Y) = B1(X, Y)
      NEXT Y
      NEXT X

30    FOR I = 1 TO NN
      FOR J = 1 TO NN
      INVM(I, J) = A(I, J) - T * B(I, J)
      NEXT J
      NEXT I

      'THE INVERSE IS CARRIED OUT HERE
      ' THE PREAMBLE OF INVERSION
      FOR I = 1 TO NN
      FOR J = 1 TO 2 * NN
      INVAM(I, J) = 0
      NEXT J
      NEXT I
      FOR I = 1 TO NN
      FOR J = 1 TO 2 * NN
      INVAM(I, J) = INVAM(I, J) + INVM(I, J)
      NEXT J
      NEXT I

      FOR I = 1 TO NN
       INVAM(I, I + NN) = INVAM(I, I + NN) + 1
      NEXT I

      FOR I = 1 TO NN
      FOR J = 1 TO NN
      INVABM(I, J) = INVAM(I, J)
      NEXT J
      NEXT I

      ZZ = 1
      'THIS IS THE PLACE FOR INVERSION PROPER
      FOR I = 1 TO NN
      OWUSS = INVAM(I, I)
40     IF OWUSS > -.0001 AND OWUSS < .0001 THEN GOTO 60
      FOR J = 1 TO 2 * NN
      INVAM(I, J) = INVAM(I, J) / OWUSS
      NEXT J
      FOR J = 1 TO NN
      IF (J = I) THEN GOTO 50
      OWUSS = INVAM(J, I)
      FOR K = 1 TO 2 * NN
      INVAM(J, K) = INVAM(J, K) - OWUSS * INVAM(I, K)
      NEXT K
50    NEXT J
      NEXT I
```

```
 ' If T > 40 Then GoTo 80
  T = T + .0001
  GOTO 30

60  ' HERE IS THE PLACE INTERCHANGE OF ROWS
    IF I + ZZ = 3 * NN THEN GOTO 80
    FOR W = 1 TO 2 * NN
    INVRM(I, W) = INVAM(I, W): INVRM(I + ZZ, W) = INVAM(I + ZZ, W)
    INVAM(I, W) = INVRM(X + ZZ, W): INVAM(I + ZZ, W) = INVRM(I, W)
    NEXT W
    OWUSS = INVAM(I, I)
    IF OWUSS = 0 THEN ZZ = ZZ + 1: GOTO 70
    Z = 1: GOTO 40
70  FOR W = 1 TO 2 * NN
    INVRM(I, W) = INVAM(I, W): INVRM(I + ZZ, W) = INVAM(I + ZZ, W)
    INVAM(I, W) = INVRM(I + ZZ, W): INVAM(I + ZZ, W) = INVAM(I, W)
    NEXT W
    IF I + ZZ = 3 * NN THEN : GOTO 80
    ZZ = ZZ + 1: GOTO 60
    'this is the end of inversion
80   PRINT
    T = T / 3.141592654# ^ 2 / P ^ 2
    PRINT "RESULT"
    PRINT "   H  =  "; T, "      P  =  "; P
90   NEXT P
End of Sub
```

SSFS RECTANGULAR PLATE

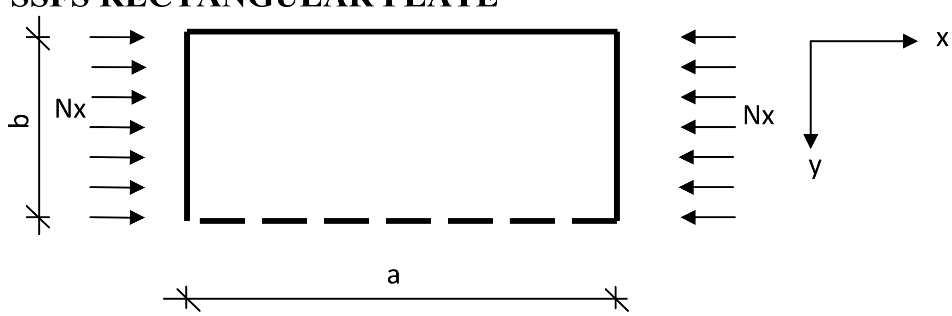

Figure 3.12: Loading arrangement for SSFS plate

For M = N = 4

Equation(159) is a 3 x 3 matrix equation that needed to be solved for the Eigen-value. BASIC program was written, as shown in PROGRAM B, to carry out the Eigen-value computation.

PROGRAM B (PROGRAM FOR SCSF PLATE)

```
              NN = 3
        REDIM AA(12, 12, 12), AANS(12, 12), ANS(12, 12)
        REDIM INVM(12, 12), INVAM(12, 12), INVRM(12, 12), INVABM(12, 12)
        REDIM A(12, 12), B(12, 12), A1(12, 12), B1(12, 12)
        CLS
        OW = 1: T = 0
         'HERE IS THE INPUT FOR INVERSION
         M = 1
10   ' PRINT "WHAT IS THE ASPECT RATIO": INPUT P: P = P * 1
        FOR P = .1 TO 1.4 STEP .1
        T = 0
        A1(1, 1) = (3.2 + 1.94286 * P ^ 2)
        A1(1, 2) = (1.92 + 1.9428 * P ^ 2)
        A1(1, 3) = (1.6 + 1.94286 * P ^ 2)
        A1(2, 2) = (1.37142 + 1.1181 * P ^ 4 + 3.49716 * P ^ 2)
        A1(2, 3) = (1.2 + 1.77156 * P ^ 4 + 3.88572 * P ^ 2)
        A1(3, 3) = (1.06667 + 2.8345 * P ^ 4 + 4.4408 * P ^ 2)
        A1(2, 1) = (1.92 + 1.9428 * P ^ 2)
        A1(3, 1) = (1.6 + 1.94286 * P ^ 2)
        A1(3, 2) = (1.2 + 1.77156 * P ^ 4 + 3.88572 * P ^ 2)
        B1(1, 1) = .3238: B1(1, 2) = .19429: B1(1, 3) = .1619
        B1(2, 2) = .13878: B1(2, 3) = .12143: B1(3, 3) = .10794
        B1(2, 1) = .19429: B1(3, 1) = .1619: B1(3, 2) = .12143
20   FOR X = 1 TO NN
        FOR Y = 1 TO NN
        A(X, Y) = A1(X, Y)
        NEXT Y
        NEXT X

        FOR X = 1 TO NN
        FOR Y = 1 TO NN
        B(X, Y) = B1(X, Y)
        NEXT Y
        NEXT X
30   FOR I = 1 TO NN
        FOR J = 1 TO NN
        INVM(I, J) = A(I, J) - T * B(I, J)
        NEXT J
        NEXT I

        'THE INVERSE IS CARRIED OUT HERE
        ' THE PREAMBLE OF INVERSION
        FOR I = 1 TO NN
        FOR J = 1 TO 2 * NN
        INVAM(I, J) = 0
        NEXT J
        NEXT I

        FOR I = 1 TO NN
        FOR J = 1 TO 2 * NN
        INVAM(I, J) = INVAM(I, J) + INVM(I, J)
```

```
        NEXT J
        NEXT I
        FOR I = 1 TO NN
         INVAM(I, I + NN) = INVAM(I, I + NN) + 1
        NEXT I
        FOR I = 1 TO NN
        FOR J = 1 TO NN
        INVABM(I, J) = INVAM(I, J)
        NEXT J
        NEXT I
        ZZ = 1
        'THIS IS THE PLACE FOR INVERSION PROPER
        FOR I = 1 TO NN
        OWUSS = INVAM(I, I)
40    IF OWUSS > -.0001 AND OWUSS < .0001 THEN GOTO 60
        FOR J = 1 TO 2 * NN
        INVAM(I, J) = INVAM(I, J) / OWUSS
        NEXT J
        FOR J = 1 TO NN
        IF (J = I) THEN GOTO 50
        OWUSS = INVAM(J, I)
        FOR K = 1 TO 2 * NN
        INVAM(J, K) = INVAM(J, K) - OWUSS * INVAM(I, K)
        NEXT K
50   NEXT J
        NEXT I
   '  If T > 40 Then GoTo 80
        T = T + .0001
        GOTO 30

60    ' HERE IS THE PLACE INTERCHANGE OF ROWS
        IF I + ZZ = 3 * NN THEN GOTO 80
        FOR W = 1 TO 2 * NN
        INVRM(I, W) = INVAM(I, W): INVRM(I + ZZ, W) = INVAM(I + ZZ, W)
        INVAM(I, W) = INVRM(X + ZZ, W): INVAM(I + ZZ, W) = INVRM(I, W)
        NEXT W
        OWUSS = INVAM(I, I)
        IF OWUSS = 0 THEN ZZ = ZZ + 1: GOTO 70
        Z = 1: GOTO 40
70    FOR W = 1 TO 2 * NN
        INVRM(I, W) = INVAM(I, W): INVRM(I + ZZ, W) = INVAM(I + ZZ, W)
        INVAM(I, W) = INVRM(I + ZZ, W): INVAM(I + ZZ, W) = INVAM(I, W)
        NEXT W
        IF I + ZZ = 3 * NN THEN : GOTO 80
        ZZ = ZZ + 1: GOTO 60
        'this is the end of inversion
80     PRINT
        T = T / 3.141592654# ^ 2 / P ^ 2
        PRINT "RESULT"
        PRINT "   H  =  "; T; "       P   =   "; P
        NEXT P
```

CCFC RECTANGULAR PLATE

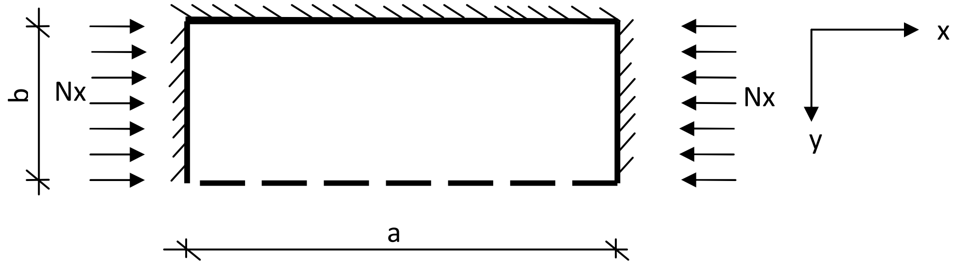

Figure 3.13: Loading arrangement for CCFC plate

For M = N = 4

Equation(163) was a 3 x 3 matrix equation that needed to be solved for the Eigen-value. BASIC program was written, as shown in PROGRAM C, to carry out the Eigen-value computation.

PROGRAM C (PROGRAM FOR CCFC PLATE)

```
                    NN = 3
      REDIM AA(12, 12, 12), AANS(12, 12), ANS(12, 12)
      REDIM INVM(12, 12), INVAM(12, 12), INVRM(12, 12), INVABM(12, 12)
      REDIM A(12, 12), B(12, 12), A1(12, 12), B1(12, 12)
      CLS
      OW = 1

       'HERE IS THE INPUT FOR INVERSION
       M = 1
10    ' PRINT "WHAT IS THE ASPECT RATIO": INPUT P: P = P * 1
      FOR P = .5 TO 1.4 STEP .1
       T = 0
      A1(1, 1) = .32 + .0127 * P ^ 4 + .1016 * P ^ 2
      A1(1, 2) = .26667 + .01904 * P ^ 4 + .1143 * P ^ 2
      A1(1, 3) = .22857 + .02539 * P ^ 4 + .12192 * P ^ 2
      A1(2, 2) = .22858 + .03808 * P ^ 4 + .13716 * P ^ 2
      A1(2, 3) = .2 + .05713 * P ^ 4 + .1524 * P ^ 2
      A1(3, 3) = .17778 + .09142 * P ^ 4 + .17416 * P ^ 2
      A1(2, 1) = .26667 + .01904 * P ^ 4 + .1143 * P ^ 2
      A1(3, 1) = .22857 + .02539 * P ^ 4 + .12192 * P ^ 2
      A1(3, 2) = .2 + .05713 * P ^ 4 + .1524 * P ^ 2
      B1(1, 1) = .00762: B1(1, 2) = .00635: B1(1, 3) = .00544
      B1(2, 2) = .00544: B1(2, 3) = .00476: B1(3, 3) = .00424
      B1(2, 1) = .00635: B1(3, 1) = .00544: B1(3, 2) = .00476
20   FOR X = 1 TO NN
      FOR Y = 1 TO NN
      A(X, Y) = A1(X, Y)
      NEXT Y
      NEXT X
```

```
        FOR X = 1 TO NN
        FOR Y = 1 TO NN
        B(X, Y) = B1(X, Y)
        NEXT Y
        NEXT X
30   FOR I = 1 TO NN
        FOR J = 1 TO NN
        INVM(I, J) = A(I, J) - T * B(I, J)
        NEXT J
        NEXT I
        'THE INVERSE IS CARRIED OUT HERE
        ' THE PREAMBLE OF INVERSION
        FOR I = 1 TO NN
        FOR J = 1 TO 2 * NN
        INVAM(I, J) = 0
        NEXT J
        NEXT I
        FOR I = 1 TO NN
        FOR J = 1 TO 2 * NN
        INVAM(I, J) = INVAM(I, J) + INVM(I, J)
        NEXT J
        NEXT I
        FOR I = 1 TO NN
         INVAM(I, I + NN) = INVAM(I, I + NN) + 1
        NEXT I
        FOR I = 1 TO NN
        FOR J = 1 TO NN
        INVABM(I, J) = INVAM(I, J)
        NEXT J
        NEXT I
        ZZ = 1
        'THIS IS THE PLACE FOR INVERSION PROPER
        FOR I = 1 TO NN
        OWUSS = INVAM(I, I)
40    IF OWUSS > -.0001 AND OWUSS < .0001 THEN GOTO 60
        FOR J = 1 TO 2 * NN
        INVAM(I, J) = INVAM(I, J) / OWUSS
        NEXT J
        FOR J = 1 TO NN
        IF (J = I) THEN GOTO 50
        OWUSS = INVAM(J, I)
        FOR K = 1 TO 2 * NN
        INVAM(J, K) = INVAM(J, K) - OWUSS * INVAM(I, K)
        NEXT K
50   NEXT J
        NEXT I
   '  If T > 40 Then GoTo 80
        T = T + .001
        GOTO 30
60    ' HERE IS THE PLACE INTERCHANGE OF ROWS
        IF I + ZZ = 3 * NN THEN GOTO 80
        FOR W = 1 TO 2 * NN
```

```
        INVRM(I, W) = INVAM(I, W): INVRM(I + ZZ, W) = INVAM(I + ZZ, W)
        INVAM(I, W) = INVRM(X + ZZ, W): INVAM(I + ZZ, W) = INVRM(I, W)
        NEXT W
        OWUSS = INVAM(I, I)
        IF OWUSS = 0 THEN ZZ = ZZ + 1: GOTO 70
        Z = 1: GOTO 40
70   FOR W = 1 TO 2 * NN
        INVRM(I, W) = INVAM(I, W): INVRM(I + ZZ, W) = INVAM(I + ZZ, W)
        INVAM(I, W) = INVRM(I + ZZ, W): INVAM(I + ZZ, W) = INVAM(I, W)
        NEXT W
        IF I + ZZ = 3 * NN THEN : GOTO 80
        ZZ = ZZ + 1: GOTO 60
        'this is the end of inversion
80    PRINT
        T = T / 3.141592654# ^ 2 / P ^ 2
        PRINT "RESULT"
        PRINT "   H  =  "; T, "       P   =   "; P
        NEXT P
```

CHAPTER FOUR
RESULTS AND DISCUSSIONS

4.1 RESULT OF SSSS THIN RECTANGULAR FLAT PLATE

The data from solution of ssss plate were presented in table 4.1

Exact solution from Iyengar (1988) was

$$(N_x)_{cr} = \left(\frac{M^2}{p^2} + \frac{P^2}{M^2} + 2\right)\frac{D\,\pi^2}{b^2}$$

The solution from present study is

$$(N_x)_{cr} = \left(\frac{1.001}{p^2} + 1.001P^2 + 2\right)\frac{D\,\pi^2}{b^2}$$

$$(N_x)_{cr} = K.\frac{D\,\pi^2}{b^2}$$

"D" means the modulus of rigidity of the plate and "b" means the edge of the plate that received the load.

The values of K for different aspect ratios are shown in table 4.1

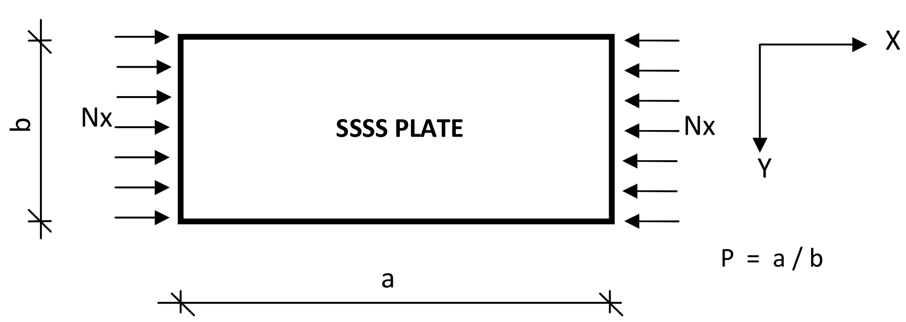

Figure 4.1: Critical Load for SSSS plate

Table 4.1: K values for different aspect ratios for SSSS Thin plate buckling

ASPECT RATIO, P = a/b	K from IYEENGAR (1988)	K from PRESENT STUDY	PERCENTAGE DIFFERENCE
0.1	102.010	102.110	0.098
0.2	27.040	27.065	0.093
0.3	13.201	13.212	0.085
0.4	8.410	8.416	0.076
0.5	6.250	6.254	0.068
0.6	5.138	5.141	0.061
0.7	4.531	4.533	0.056
0.8	4.203	4.205	0.052
0.9	4.045	4.047	0.051
1	4.000	4.002	0.050

The average percentage difference between the solution from Iyenger (in this case exact) and the present study according to table 4.1 was 0.069%. It would also be noticed that the closeness of the two solutions improved as the aspect ratio increases from 0.1 to 1. This meant that the solution from this present study was a very close approximate of the exact solution. Hence, the assumed deflection function was very close to the exact shape function. However, it was shown that the solution was upper bound solution.

4.2 RESULT OF CCCC THIN RECTANGULAR FLAT PLATE

The comparison of the data from this present study and Iyengar (1988) solution was presented in table 4.2. In this solution, Iyengar assumed trigonometric series in formulating the shape function. He got the solution of plate at first buckling mode as

$$(N_x)cr = \left(\frac{4}{P^2} + 4P^2 + 2.667\right)\frac{\pi^2 D}{b^2}$$

The solution of this present study is

$$(N_x)cr = \left(\frac{4.255}{P^2} + 4.255P^2 + 2.428\right)\frac{\pi^2 D}{b^2}$$

$$(N_x)_{cr} = K.\frac{D\,\pi^2}{b^2}$$

The values of K for different aspect ratios are shown in table 4.2

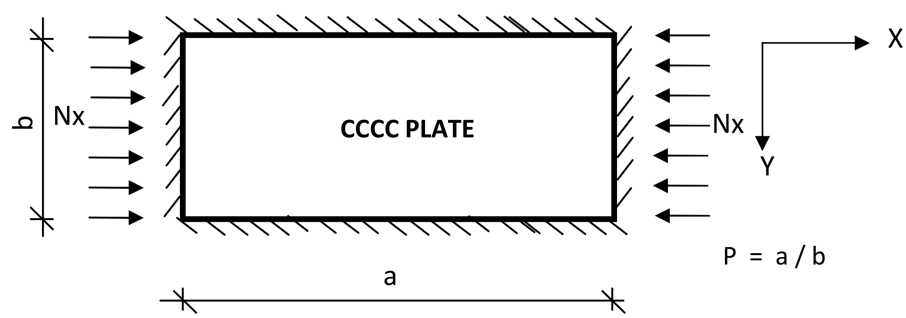

Figure 4.2: Critical Load for CCCC plate

Table 4.2: K values for different aspect ratios for CCCC Thin plate buckling

ASPECT RATIO, P = a/b	K from IYEENGAR (1988)	K from PRESENT STUDY	PERCENTAGE DIFFERENCE
0.1	402.707	424.970	5.528
0.2	102.827	108.222	5.247
0.3	47.471	49.753	4.806
0.4	28.307	29.510	4.251
0.5	19.667	20.384	3.647
0.6	15.218	15.685	3.069
0.7	12.790	13.121	2.583
0.8	11.477	11.734	2.235
0.9	10.845	11.066	2.038
1	10.667	10.878	1.978

The present study gave a solution to CCCC thin plate buckling, which has average percentage difference with the solution from Iyenger (in this case solution from trigonometric function) as 3.538%. This is an upper bound approximation. It would also be noticed that the closeness of the two solutions improves as the aspect ratio increases from 0.1 to 1, given the corresponding percentage difference as from 5.528% to 1.978%. This meant that the solution from this present study was a close approximate of the exact solution if it is assumed that solution from trigonometric functions are close to exact solution. Hence, the assumed deflection function was close to the exact shape function.

On the other hand, Levy (1942) used an infinite series to get the solution for a square plate (that is aspect ratio is 1) as

$$(N_x)cr = 10.07\frac{\pi^2 D}{b^2}$$

The corresponding solution from this study was

$$(N_x)cr = 10.878\frac{\pi^2 D}{b^2}$$

This was an upper bound solution in comparison with Levy's solution. The average percentage difference was 8.024%. It would be nice to know that Iyenger's solution differs from Levy's by 5.929%. These differences are quite acceptable in statistics as being close. It went further to affirm the good approximation of the shape function using finite power series. However, it is not yet certain which of the solutions (Iyenger or Levy) is closer to the exact solution.

4.3 RESULT OF CSCS THIN RECTANGULAR FLAT PLATE

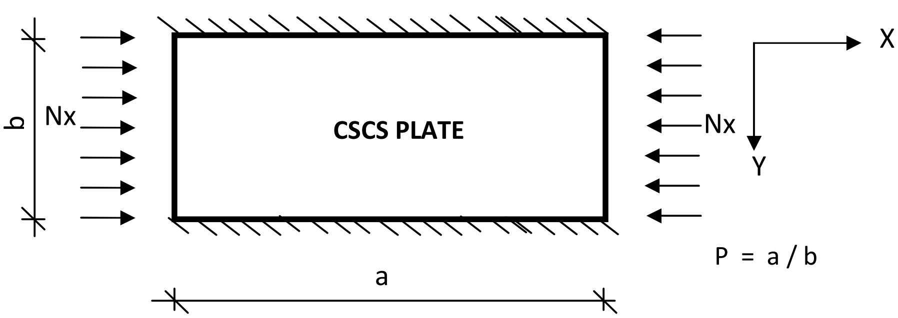

Figure 4.3: Critical Load for CSCS plate

Iyengar (1988) obtained a general solution for this plate by introducing a trigonometric function as shape function into the energy equation. He got

$$(N_x)cr = \left(\frac{M^2}{P^2} + \frac{5.333P^2}{M^2} + 2.667\right)\frac{\pi^2 D}{b^2}$$

The solution from the present study was

$$(N_x)cr = \left(\frac{1.001}{P^2} + 5.165P^2 \ + 2.428\right)\frac{\pi^2 D}{b^2}$$

$$(N_x)_{cr} = K.\frac{D\,\pi^2}{b^2}$$

The values of K for different aspect ratios are shown in table 4.3

Table 4.3: K values for different aspect ratios for CSCS Thin plate buckling

Aspect Ratio, P = A/B	K From Present Study	K From Iyeengar (1988)	K From Timoshenko (1936), and Heck and Ebner (1936)	Percentage difference between Present and Iyenger	Percentage difference between Present and Timoshenko & Heck and Ebner	Percentage difference between Iyenger and Timoshenko & Heck and Ebner
0.1	102.580	102.720	*	-0.137	*	*
0.2	27.660	27.880	*	-0.792	*	*
0.3	14.015	14.258	*	-1.704	*	*
0.4	9.511	9.770	9.440	-2.657	0.748	3.499
0.5	7.723	8.000	7.680	-3.462	0.563	4.170
0.6	7.068	7.365	7.050	-4.029	0.255	4.463
0.7	7.002	7.321	7.000	-4.361	0.024	4.586
0.8	7.298	7.643	7.300	-4.514	-0.032	4.693
0.9	7.847	8.221	*	-4.547	*	*
1	8.594	9.000	7.680	-4.511	11.901	17.188

* Means not available

Comparison of the two solutions was as shown in table 4.3. The average percentage difference between the solutions from the present study and Iyengar, according to table 4.3, is -3.0714%. On the other hand, the solution from the present study was compared with the solution by Timoshenko (1936) and Heck and Ebner (1936). The average percentage difference is 2.24 %. This is within the range of acceptance in statistics (Nwaogazie) 1999). It was interesting that both Iyengar's (1988), Timoshenko's (1936) and Heck and Ebner's (1936) solutions were not exact solutions but approximate. The exact solution should be somewhat around these solutions (including the solution of this present study). Another interesting thing was that for aspect ratios of between 0.4 and 0.8, the solution of this present study had an average percentage difference of 0.31% with the solutions of Timoshenko, and Heck and Ebner. However, it would be noticed that for aspect ratio of 1, the percentage differences between the present study, and Timoshenko, and Heck and Ebner are quite high (11.901 %).

4.4 RESULT OF SCSC THIN RECTANGULAR FLAT PLATE

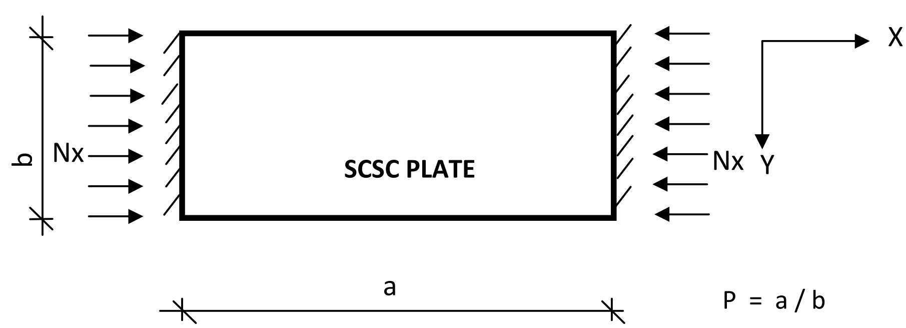

Figure 4.4: Critical Load for SCSC plate

Data from this present study, Iyengar (1988) and Tmoshenko (1936) of SCSC plate were presented in table 4.4. The general solution as obtained in this study is

$$(N_x)cr = \left(\frac{4.244}{P^2} + 0.822P^2 + 1.994\right)\frac{\pi^2 D}{b^2}$$

$$(N_x)cr = \frac{K.D\pi^2}{b^2}$$

The values of K for different aspect ratios are shown in table 4.4

Table 4.4: K values for different aspect ratios for SCSC Thin plate buckling

Aspect ratio	K from Iyenger I	K from Timoshenko T	K from Present Study P	Difference		% Difference	
				P - I	P - T	Between P and I	Between P and T
0.6	13.381	13.37	14.079	0.698	0.709	5.216	5.303
0.8	8.73	8.73	9.151	0.421	0.421	4.822	4.822
1	6.75	6.74	7.06	0.31	0.32	4.593	4.748

Legend: I means Iyengar; T means Timoshenko; P means present study

The average percentage differences between the solution of the present study and Iyengar (1988), and the present study and Timoshenko (1936) were 4.88% and 4.96% respectively. This percentage differences were quite acceptable as not being significant in statistics.

4.5 RESULT OF CSSS THIN RECTANGULAR FLAT PLATE

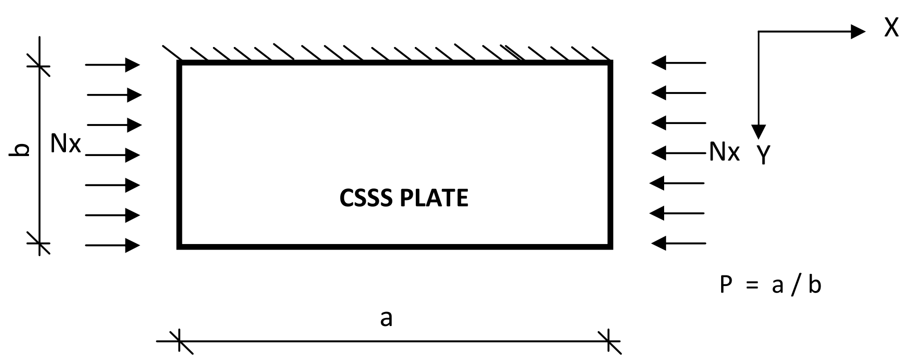

Figure 4.5: Critical Load for CSSS plate

This study obtained a general solution for this plate as

$$(N_x)cr = \left(\frac{1.001}{P^2} + 2.45P^2 + 2.304\right)\frac{\pi^2 D}{b^2}$$

$$(N_x)cr = \frac{K.D\pi^2}{b^2}$$

Available reference solutions were from FOK (1980) and Michelutti (1976). Fok used finite difference method to obtain solution for a square plate (aspect ratio of 1) as:

$$(N_x)cr = 5.4\frac{\pi^2 D}{b^2}$$

Michelluti used Runge-Kutta method to obtain solution of a plate with aspect ratio of 0.79 as:

$$(N_x)cr = 5.41\frac{\pi^2 D}{b^2}$$

Table 4.5: K values for different aspect ratios for CSSS Thin plate buckling

ASPECT RATIO, P = a/b	K from PRESENT STUDY P	K from Fok (1980) F	K from Michelluti (1976) M	PERCENTAGE DIFFERENCE between P and F	PERCENTAGE DIFFERENCE between P and M
0.79	5.44	*	5.41	0.55	*
1	5.76	5.4	*	*	6.25

* Means not available

Legend: F means Fok; M means Michelluti; P means present study

With these solutions, the percentage difference between the solution of the present study and that of FOk was 6.25%. The average percentage difference with Michelutti was 0.55%. These differences implied that the solution from this present study was close to both solutions from Fok and Michelluti. However, in both cases the solution of this study was upper bound though the reference solutions (Fok and Michelluti) were not exact solutions. Hence, the approximate shape function was well founded, and was good approximation of the exact shape function.

4.6 RESULT OF SCSS THIN RECTANGULAR FLAT PLATE

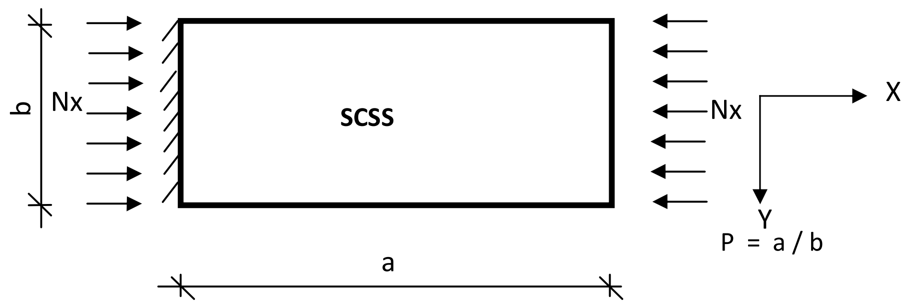

Figure 4.6: Critical Load for SCSS plate

No reference solution was cited at the time of documenting the literature works. Hence, there was no basis for comparison of solutions. However, this study came up with a general solution of

$$(N_x)cr = \left(\frac{2.128}{P^2} + 0.869P^2 + 2\right)\frac{\pi^2 D}{b^2}$$

4.7 RESULT OF CCSC THIN RECTANGULAR FLAT PLATE

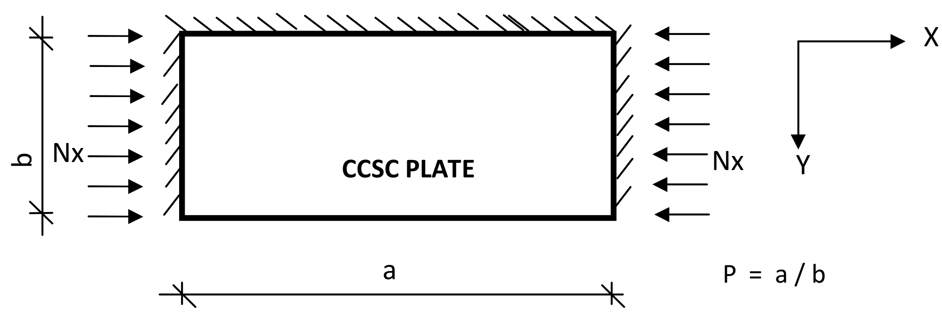

Figure 4.7: Critical Load for CCSC plate

No reference solution (of aspect ratio from 0.1 to 1) was cited at the time of documenting the literature works. Hence, there was no basis for comparison of solutions. However, this study came up with a general solution of

$$(N_x)cr = \left(\frac{4.25}{P^2} + 3.677P^2 + 2.303\right)\frac{\pi^2 D}{b^2}$$

4.8 RESULT OF SSFS THIN RECTANGULAR FLAT PLATE

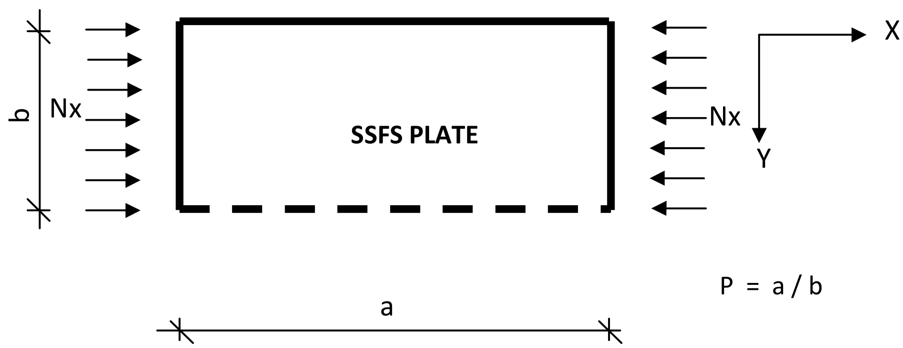

Figure 4.8: Critical Load for SSFS plate

A critical look at table 4.6 revealed that the average percentage difference between the result from Timoshenko and the present study is 4.14%. This difference is acceptable in statistics (Nwaogazie, 1999) as any difference may be attributed to approximations introduced to the process from both Timoshenko and the present study. Both of these results are approximate. None is the exact solution. However, the exact solution is suspected to be near the neighborhood of these two sets of solution.

Table 4.6: K values for different aspect ratios for SSFS Thin plate buckling

Aspect Ratio	K From Timoshenko (1936)	K Present study	difference	% difference

0.5	4.401	4.515	0.114	2.59
1	1.44	1.522	0.082	5.69

4.9 RESULT OF CSFS THIN RECTANGULAR FLAT PLATE

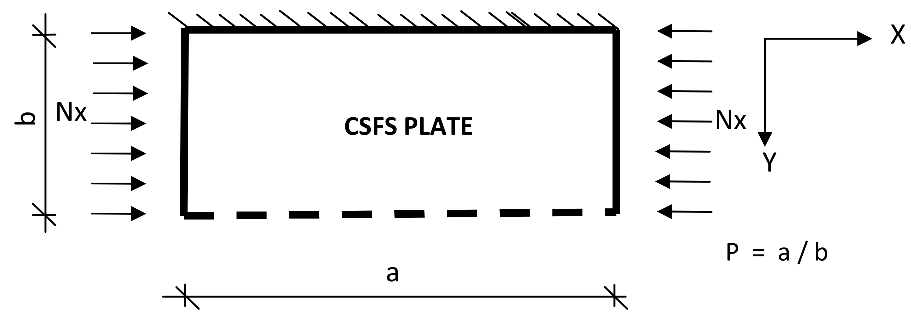

Figure 4.9: Critical Load for CSFS plate

Average percentage differences of 14.48% and 14.12% of the result of this study with those of Iyengar and Timoshenko respectively as can be seen in tables 4.7 and 4.8 are high. However they are acceptable as being close in statistics (Nwaogazie 1999). A common thing about these two sets of results is that they are all approximate solutions. But the exact solution is suspected to be in the neighborhood of the two. This is so because the two solutions, though not the same with each other, but are closer to each other.

Table 4.7: K values for different aspect ratios for CSFS Thin plate buckling plate.

Aspect Ratio	K from Iyengar	K from Present study	Difference	% difference
0.2	25.35	18.47	6.88	27.14

0.4	6.34	6.09	0.25	3.94
0.8	2.16	2.42	0.28	12.96

Table 4.8: K values for different aspect ratios for CSFS Thin plate buckling

Aspect Ratio	K from Timoshenko (1936)	K from Present study	Difference	% difference
1	1.7	1.94	0.24	14.12

4.10 RESULT OF CCFC THIN RECTANGULAR FLAT PLATE

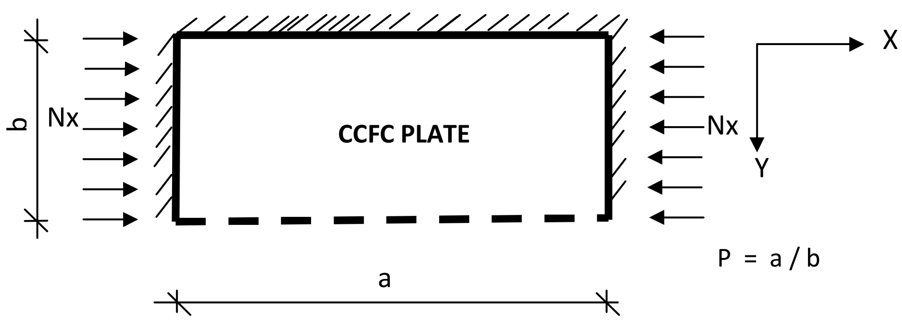

Figure 4.10: Critical Load for CCFC plate

No reference solution was obtained during the time of documenting the literature review. Hence, there is no basis to compare the solution from this study. However, the solution were made available in this work. The values of K for different aspect ratios are as show in table 4.10

Table 4.10: values for different aspect ratios for CCFC Thin plate buckling

Aspect Ratio	K
0.1	159.16
0.2	54.24
0.3	35.41

0.4	27.29
0.5	17.73
0.6	12.54
0.7	9.42
0.8	7.41
0.9	6.05
1	5.08

4.11 CONVERGENCE TEST OF THE DISPLACEMENT FUNCTION

To ascertain that the displacement function of equation 102 really converged at M = N = 4, this test become necessary. For convenience, equation 102 is repeated here as $w = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} I_m J_n\ x^m . y^n$ (102)

It could be recalled that the work presented in this project is based on equation 102, which was truncated at M = N = 4. To achieve this, CSCS plate and CCSS plate were used. The values of K were determined for the two plates of different boundary conditions (CSCS and CCSS) when M = N = 4 and M = N = 5

4.11.1 CONVERGENCE OF DISPLACEMENT FUCTION FOR CSCS

This convergence is presented in table 4.11. It could be seen from the table that the maximum percentage difference between K values for M=N=4 and M=N=5 is 1.11 for the plate with aspect ratio of 0.1. This value is not significant at all. However, the difference could be attributed to round off errors involved in the computation. With this, it becomes apparent that the displacement function converges at M=N=4. Extra computational effort involved in M=N=5 did not yield improved result, hence not necessary.

Table 4. 11: comparison of K values for M=N=4 and M=N=5 for CSCS plate

ASPECT RATIO	K value for M=N=5	K value for M=N=4	KM4 - KM5	PERCENTAGE DIFFERENCE
0.1	101.44	102.58	1.14	1.111328
0.2	27.36	27.66	0.3	1.084599
0.3	13.89	14.02	0.13	0.927247
0.4	9.44	9.51	0.07	0.736067
0.5	7.68	7.72	0.04	0.518135
0.6	7.04	7.07	0.03	0.424328
0.7	6.99	7	0.01	0.142857
0.8	7.29	7.3	0.01	0.136986
0.9	7.85	7.85	0	0
1	8.6	8.59	-0.01	-0.11641
1.1	9.51	9.5	-0.01	-0.10526
1.2	10.56	10.56	0	0
1.3	11.75	11.75	0	0

4.11.2 CONVERGENCE OF DISPLACEMENT FUCTION FOR CCSS

This convergence is presented in table 4.12. It could be seen from the table that the maximum percentage difference between K values for M=N=4 and M=N=5 is 0.878 for the plate with aspect ratio of 0.3. This value is not significant at all. However, the difference could be attributed to round off errors involved in the computation. With this, it becomes apparent that the displacement function converges at M=N=4. Extra computational effort involved in M=N=5 did not yield improved result, hence not necessary.

Table 4. 12: comparison of K values for M=N=4 and M=N=5 for CCSS plate

ASPECT RATIO	M=N=5	M=N=4	M4 - M5	PERCENTAGE DIFFERENCE
0.1	213.51	215.12	1.61	0.748419
0.2	55.18	55.59	0.41	0.737543
0.3	25.96	26.19	0.23	0.878198
0.4	15.84	15.94	0.1	0.627353

0.5	11.28	11.35	0.07	0.61674
0.6	8.93	8.98	0.05	0.556793
0.7	7.65	7.69	0.04	0.520156
0.8	6.96	6.99	0.03	0.429185
0.9	6.63	6.65	0.02	0.300752
1	6.54	6.56	0.02	0.304878
1.1	6.62	6.64	0.02	0.301205
1.2	6.83	6.85	0.02	0.291971
1.3	7.15	7.16	0.01	0.139665

4.12 RESULT OF ITERATION MATRIX INVERSION METHOD

The resulting lowest eigenvalus from the above five problem from using the Q BASIC program are as shown in Table 4.13. When these lowest Eigen-values are substituted back into their respective problems, and the determinants of the problems calculated, the resulting values of the determinants are as shown in Table 4.14. It is a common knowledge that the determinant of an Eigen-value matrix is zero when the exact Eigen-value is substituted into it. Hence, if the Eigen-values in table 4.13 were exact or approximate Eigen-values of matrices 1, 2, 3, 4 and 5 then the determinants, upon substituting the Eigen-values into the matrices, would be exactly or approximately equal to zero. A close at table 4.14 would reveal that the determinants from Iteration-Matrix Inversion (I-MI) method were approximately very close to zero. Looking at table 4.14 again shows that the determinant for matrix 2 from power method (James, Smith and Wolford, 1977) is far from being zero. From the foregoing, it is clear that I-MI method was efficient in the convergence to exact solutions of Eigen-values.

Table 13: Results of Eigen-value Problems

Problem	Lowest Eigen-value from Iteration-Matrix Inversion method	Lowest Eigen-value from Reference

1	1	1
2	0.031	1.98
3	15.113	No reference
4	39.507	No reference
5	47.24	No reference

Table 4.14: Determinants of Eigen-value Problems

Problem	Determinant from Iteration-Matrix Inversion method	Determinant from Reference
1	0	0
2	-0.0000032	-5.50815
3	0.00000567	No reference
4	0.0000000045	No reference
5	0.0000000000021741	No reference

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.1. CONCLUSIONS

Based on the result of this research, the following conclusions could be drawn:

i. Taylor series (polynomial) is satisfactory in approximating the deformed shape of thin rectangular plates of various boundary conditions.

ii. Taylor series (polynomial) truncated at M=N=4 could easily be used to approximate the deformed shape of thin rectangular plates of various boundary conditions.

iii. A direct variation principle (based on Raleigh-Ritz method) could be used in confidence to satisfactorily analyze real time rectangular thin plates of various boundary conditions under in- plane loadings.

iv. The results obtained herein are very close to the results obtained by previous research works that used different methods of analysis.

v. The use of I-MI method is not only efficient in convergence but also capable of handling Eigen-value problems that uses consistent mass or stiffness matrices. The lowest Eigen-values from the use of I-MI method is very reliable. I-MI method can be used for problems whose matrices are of n X n order ($2 \leq n \leq \infty$). The limitation to the use of I-MI is the memory capacity of the computer to be employed.

5.2 RECOMMENDATIONS

The following recommendations were made:

i. Future research work should consider using Taylor series displacement function in the direct variational methods other than Raleigh-Ritz method in buckling analysis of thin plates.

ii. Future studies should use the truncated Taylor series derived herein and Raleigh-Ritz method to analyze plate under transverse loading.

iii. Future studies should try to use Taylor series truncated beyond M=N=5 to see if it could improve on the result for plate with one of its edges free of support.

REFERENCES

Arbenz, P. and Geus, R. (2005). Multilevel preconditioned iterative eigensolvers for maxwell Eigen-value problems. Applied numerical mathematics. Vol. 54 , issue 2 .pp 107 - 121

Arnoldi, W. E. (1951). The principle of minimized iteration in the solution of the matrix Eigen-value problem, Quarterly of Applied Mathematics, vol. 9, pp. 17– 29.

Arthurs, A. M. (1975). Calculus of variations. London: Routledge & Kegan Paul.

Attaalla, S. A. (2002). Inelastic buckling strength of unsymmetrical tapered plates. Advances in structural Engineering, Vol. 5, No. 3 (August), pp.165-171.

Audoly, B., Roman, B. And Pocheau A. (2002). Secondary buckling patterns of a thin plate under in-plane compression. The European physical Journal B-Condensed Matter and Complex systems, vol. 27, No. 1 (May).

Azhari, M. and Bradford, M. A. (1994). Local buckling by complex finite strip method using bubble functions. Journals of Engineering mechanics, Vol. 120, No. 1 (January)

Bai, Z. Demmel, J., Dongarra, J. Ruhe, A., and van der Vorst, H. (2000). Templates for the Solution of Algebraic Eigen-value Problems - A Practical Guide, SIAM, Philadelphia, PA,

Barrett, R., Berry, M., Chan, T. F., Demmel, J., Donato, J., Dongarra, J., Eijkhout, V., Pozo, R., Romine, C. and van der Vorst, H. A. (1994). Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods, 2nd Edition, SIAM, Philadelphia, PA,

Basu, A. K. And Chapman, J. C. (1966). Large deflection behaviour of transversely loaded rectangular orthotropic plates. Proc. ICE 35, 79-110.

Bergmann, S. U. And Reissner, H. (1932). Neuere probleme aus der flugzeugstatik über die knickung von rechteckigen platten schubbeanspruchung. ZPM, vol. 23, 6-12 (in German).

Bernoulli, J. (1789).Eessai theoretique sur les vibrations des plaques elastiques rectangulaires et libres. Nova Acta, vol. 5. St. Petersburg.

Bleich, F. (1952). Buckling strength of metal structures. New York: McGraw-Hill.

Bollenrath, F. (1931) Wrinkling phenomena of thin flat plates subjected to shear stresses. NACA, Tech. Memo., No. 601.

Boyce, W. E. and Diprima, R. C. (1977). Elementary differential equations and boundary value problems 3rd. Ed. New York: John Wiley.

Bullson, P.S. (1970). The Stability of Flat plates. London: Chatto & Windus.

Chatelin, F. (1993). Eigen-values of matrices, JohnWiley&SonsLtd,Chichester,West Sussex, England, Originally published in two separate volumes by Masson, Paris: Valeurs propres de matrices (1988) and Exercises de valeurs propres de matrices (1989).

Chen, J., Xu, Y. and Zou, J. (2010). An adaptive inverse iteration for Maxwell Eigen-value problem based on edge elements. Journal of Computational Physics, Vol. 229, Issue 7, 1 April 2010, Pp. 2649-2658

Cook, R. D., Malkus, D. S. And Plesha, M. E. (1989). Concepts and applications of finite element analysis. New York: John Wiley.

Courant, R. and Hilbert, D. (1953). Methods of mathematical physics. Interscience, Vol. 1.

Cox, H.L. (1933). Summary of the present state of knowledge regarding sheet metal construction. R & M, No. 1553, 423-442.

Davies, G. A. O. (1982). Virtual work in structural analysis. Chichester: John Wiley.

Demmel, J. W. (1997). Applied Numerical Linear Algebra, SIAM, Philadelphia, PA

Donnell, L. H. (1976). Beams, plates and shells. New York: McGraw-Hill.

Eccher, G., Rasmussen, K. J. R. and Zandonini, R. (2007). Shear locking in isoparametric spline finite strips. School of Civil Engineering University of Sydney., Research Report No. R876, (February)

Eccher, G., Rasmussen, K.J.R. and Zandonini, R. (2007). Geometric nonlinear isoparametric spline finite strip analysis of perforated thin-walled structures. School of Civil Engineering of Sydrey , Research Report No. R880 (February).

Edwards, P. (2004). MacLaurin series. Bournemouth University UK. (http://mathinsite.bmth.ac.uk./html.applets.html)

Elsgolts, L. (1980). Differential Equations and the calculus of variations. 3rd Ed. Moscow: Mir Publishers..

Fok, C. D. (1980). Effects of initial imperfections on the elastic post-buckling behaviour of flat plates. Ph.D. Thesis, Monash University.

Freitag, M. (2007). Inner-outer Iterative Methods for Eigen-value Problems - Convergence and Preconditioning. PhD Thesis submitted to University of Bath

Geradin, M. (1980). Variational Methods of Structural Dynamics and their Finite Element Implementation. In J. Donea (Ed.), Advanced Structural Dynamics (pp.1-42)

Greenberg, M. (1998). Advanced engineering mathematics. 2nd. Ed. Englewood Cliffs. N.J.: Prentice Hall.

Greenstadt, J. (1960). Mathematical Methods for Digital Computers. Vol. 1, ed. A. Ralston and H. S. Wilf (New York: John Wiley & sons). Pp. 84-91.

Hartmann, F. (1991)., Static analysis of plates. In: Boundary Element Analysis of Plates and Shells (ed. D.E. Beskos), Springer-Verlag, Berlin, pp. 1–35

Heck, O. S. and Ebner, H. (1936). Methods and formulas for calculating the strength of plate and shell construction as used in Airplane Design. National Advance comm. Aeron., Tech. Memo. 785.

Hochstenbach, M. E. and Notay, Y. (2004). The Jacobi-Davidson method, GAMM Mitt.,

Huber, M. T. (1914). Die Grundlagen einer rationeller Berechnung der kreuz-waise bewehrten eisenbeton platten. Zeitscher. Österreichischen ing. und arch. Verein 66, 577.

Hughes,T.J.R.(1987). TheFiniteElementMethod,Prentice-Hill, Englewood Cliffs, New Jersey

Ishizu, K. (1968). Variational methods in elasticity and plasticity. Oxford: Pergamon press.

Iyengar, N. G. (1988). Structural stability of columns and plates. Chichester : Ellis Horwood.

James, M. L., Smith, G. M. and Wolford, J. C. (1977). Applied Numerical Methods for Digital Computation: with FORTRAN and CSMP. Harper and Row Publishers, New York.

Karman, V. T. (1970). Festigkeits probleme im maschinenbau. Encyklopädie der mathematischen wissenschaften. Vol. 4.B.G. Teubner, Leipzig.

Key, S. W. (1980). Transient Response by Time Integration: Review of Implicit and Explicit Operators. In J. Donea (Ed.), Advanced Structural Dynamics (pp.

Key, S. W. and Krieg, R. D. (1972) Transient Shell Response by numerical Time Integration. 2nd US-JAPAN Seminar on Matrix Methods in Structural Analysis. Berkeley, California.

Kirchoff, G. R. (1977). Vorlesungen über mathematische physic, mechanik (2nd ed), p.450.

Lanczos, C. (1949). The Variational Principles of mechanics. 4th ed. Toronto: University of Toronto press.

Lanczos, C. (1950). An iterative method for the solution of the Eigen-value problem of linear differential and integral operators, Journal of Research of the National Bureau of Standards, vol. 45 , pp. 255–282.

Langhaar, H. L. (1962). Energy methods in applied mechanics. New York: John Wiley.

Leipholz, H. (1977). Direct variational methods and Eigen-value problems in engineering. Leyden: Noordhoff Int. Publishers.

Levy, S. (1942). Bending of rectangular plates with large deflection. NACA ,teh. Note No. 846.

Levy, S. and Greenman, S. (1942). Bending with large deflection of a clamped rectangular plate with length-width ratio of 1.5 under normal pressure. NACA, tech. Note No. 853.

Mansfield, E. H. (1964). The bending and stretching of plates. New York: Macmillan.

Marguerre, K. (1938). Zur theorie der gekrümmmter platte grosser formänderung. Proc. Fifth int. congress Appl. Mech., P.93.

Maulbetsch, J. L. (1937). Buckling of compressed rectangular plates with built-in edges. ASME Journal of Applied mechanics, vol. 4. No. 2 (June).

Michelutti, W. (1976). Stiffened plates in combined loading. Ph. D. Thesis, Monash University.

Mikhlin, S. G. (1964). Variational methods in mathematical physics. New York: Macmillan.

Mindlin, R. D. (1951). Influence of rotary inertia and shear in flexural motion of isotropic, elastic plates. J. Appl. Mech., 18(March), 1031-1036.

Moheit, W. (1942). Schubbeulung rechteckiger platten mit einges panten rändern. Stahlbau, vol. 13, 39-44 (in German)

Murray, N.W. (1984). Introduction to the theory of thin-walled structures. Oxford: Clarendon press.

Navier, L. (1823). Bull. Soc. Phil. Math., Paris.

Ortega, J. (1967). Mathematical Methods for Digital Computers. Vol. 2, ed. A. Ralston and H. S. Wilf (New York: John Wiley & sons). Pp. 94-115.

Owen, D. R. J. (1980). Implicit Finite Element Methods for The Dynamic Transient Analysis of Solids with Particular Reference to Non-linear Situations. In J. Donea (Ed.), Advanced Structural Dynamics (pp. 123-152)

Paz, M. (1980). Structural Dynamics: Theory and Computation. Litton Education Publishers, New York.

Plaut, R. H. And Guran, A. (1994). Buckling of plates with stiffening elastically restrained edges. journal of engineering mechanics, Vol.120 No.2 (February).

Quillen, P. and Ye, Q. (2010). A block inverse-free preconditioned Krylov subspace method for symmetric generalized Eigen-value problems. Journal of Computational and Applied Mathematics. Vol. 233 , Issue 5. Pp. 1298-1313

Rao, S. S. (1989). The finite element method in engineering. 2nd Ed. Oxford: Pergamon press.

Rayleigh, J. (1945). Theory of sound. New York: Dover Publications.

Reddy, J.N. (1984). Energy and variational methods in applied mechanics. New York : John Wiley..

Reddy, J.N. (2002). Energy principles and variational methods in applied mechanics.

Reissner, E. (1945). The effect of transverse shear deformation on the bending of elastic plates. J. Appl. Mech, 12(June), 69-76.

Richards, T. H. (1977). Energy methods in stress analysis. Chichester: Ellis Horwood.

Ritz, W. (1908). Über eine neue method zur Lösung gewisser variations probleme der mathematischen Physik. Zeits chrift für Reine und Angewandte Mathematik. 135, 1-61.

Saint Venant (1883). Discussion in theorie de l'elasticite des corps solide, clebsch, P.704.

Seydel, E. (1933). Ûber das ausbeulen von recteckige isotroper oder othorgonal anithrtropen platten beir schubbean-spruchung. Ing. Archiv, vol. 4, 169-191(in German)

Shames, I. H. and Dym, C. L. (1985). Energy and finite element methods in structural mechanics. Ishington, D C.: Hemisphere.

Sheikh, A. H., Haldar, S. and Sengupta, D. (2004). Free flexural vibration of composite plates in different situations using a high precision triangular element. Journal of Vibration and Control, Vol. 10, No. 3, 371-386

Sleijpen, G. L. G. and van der Vorst, H. A. (1996). A Jacobi-Davidson iteration method for linear Eigen-value problems, SIAM J. Matrix Anal. Appl., 17 , pp. 401–425.

Stein, M. And Neff, J. (1947). Buckling Stressess of simply supported rectangular flat plates in shear. NACA, Tech Note No. 1222.

Stroud, K. A. (1982). Engineering mathematics 2nd Ed. London: Macmillan.

Szilard, R. (1974). Theory and analysis of plates. Upper Saddle River, N. J.: Prentice Hall.

Szilard, R. (2004). Theories and Applications of plate analysis (Classical, Numerical and Engineering Methods). New Jersey: John Wiley & Sons.

Thomas, G. B. (Jr) and Finney, R. L. (1996). Calculus and analytic geometry (9th ed.). Addison Wesley. ISBN. 0-201-53174-7.

Timoshenko (1913). Sur la stabilite des systems elastiques. Ann ponts et chausses.

Timoshenko, S. (1936). Theory of elastic stability. Engineering societies monograph, McGraw-hill Book Company.

Timoshenko, S. P. (1907). Bull Polytech.Inst.Kiev.(in Russian). Published later as: Einige stabilitäts probleme der Elasticitats theorie. Zeitsch. Für Math. Und physic [1910], 337-85.

Timoshenko, S. P. (1921). Über die stabilität versteifter platten. Eisenbau, vol. 12, 147 – 163.

Timoshenko, S. P. And Gere, J.M. (1961). Theory of elastic stability. Kogakusha: McGraw-hill.

Timoshenko, S. P. And Woinowsky-krieger, S. (1970). Theory of plates and shells (2nd Ed.). Singapore: Mc Graw-Hill Book Co. P.379.

Tranter, C. J. and Lambe, C. G. (1973). Advanced level mathematics: pure and applied. 3rd ed. London:The English Universities press.

Trefethen, L. N. and Bau, D. I. (1997). Numerical Linear Algebra, SIAM, Philadelphia, PA,

Ugural, A. C. (1999). Stresses in plates and shells (2nd Ed.). Singapore: McGraw-hill.

Venkateswara Rao, G., Rajasekhara Naidu,R. and kanak Raju, K. (1994). Postbuckling of moderately thick circular plates with edge elastic restraint. Journal of Engineering Mechanics vol. 120.

Venkateswara RGO, G. and kanak Raju, K. (1983). A reinvestigation of post buckling behaviour of elastic circular plates using a simple finite element formulation. Comp. and struct., 17(2), 233-235.

Ventsel, E.S. (1997), An indirect boundary element method for plate bending analysis, International J Numer Meth Eng, 9, pp. 1597–1611

Ventsel, E. And Krauthammer, T. (2001). Thin Plates and Shells: Theory, Analysis and Applications. New York: Marcel Dekker

Vinson, J. R. (1974). Structural mechanics: The behaviour of plates and shells. New York: John Willey.

Wang, P.Ch. (1969), Numerical and Matrix Methods in Structural Mechanics, John Wiley and Sons, New York.

Wang, Xin wei; Wang, Xinfeng; Shi, Xudong (2006). Differential quadrature buckling analysis of rectangular plates subjected to non-uniform distributed in-plane loadings. Thin-walled Structures Coden Tisde, vol. 44, No. 8, Pp837-843.

Way, S. (1938). Uniformly loaded clamped rectangular plates with large deflections. Proc. fifth intl. congr. of Appl. Mech., PP123-8.

Wilkinson, J. H. (1965). The Algebraic Eigen-value Problem, Oxford University Press, Oxford, UK

Ye, Jianqiao. (1994). Large deflection of imperfect plates by iterative BE-FE method. Journal of Engineering Mechanics, Vol. 120, No. 3 (March)

Yu, C. and Schafer, B. W. (2007). Effect of longitudinal stress gradients on elastic buckling of thin plates. Journal of Engineering Mechanics, Vol. 133, Issue 4 (April), pp.452-463.

Zhu, yi and Wilkinson, Tim (2007). Finite element analysis of structural steel elliptical hollow sections in compression. School of Civil Engineering University of Sydney, Research report No. R874, February.

INDEX

A

B

C

D

E

F

G

H

I

K

L

M

N

P

R

S

T

U

V

W

X

Y

Z