CONTROLLABILITY RESULTS FOR NON-LINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

BY

PAUL ANAETO ORAEKIE

B. Sc. (HONS) MATHEMATICS (UNN),

M. Sc. MATHEMATICS (N A U)

REG. N0: 2005 45 48 658.

A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE (*DOCTOR OF PHILOSOPHY*) *PhD* IN MATHEMATICS,

NOVEMBER, 2012.

CERTIFICATION

This is to certify that **Oraekie, Paul Anaeto**, a postgraduate student of the Department of Mathematics and Computer Science, Feder-al University of Technology, Owerri, Nigeria and with **Reg.N0: 20054548658** has satisfactorily completed the requirements for course work and research work for the award of the degree of Doctor of Philosophy **(PhD)** in Mathematics.

The work embodied in this thesis is original and has not been submitted in part or full for any other degree of this or any other university.

………………………………….. ……………………………...
Oraekie, Paul Anaeto Date
Student

………………………………….. ……………………….……..
D r. C.A Nse Date
Co-Supervisor

…………………………..….. ……………………..……..
Dr. Inyama, S. C. Date
Co-Supervisor

………………………………… ……………………………..
Dr. V.A Iheagwam Date
Principal Supervisor

………………………………… ……….……………………
Prof. I. S. Iwueze Date
Head, Department of Mathematics

……………………………………… ……………………………….
Prof.F.O.U.Osuala Date
Dean School of Science

…………………........................... …………………………..
Engr.Prof.K.B.Oyoh Date
Dean, Postgraduate School

…………………........................... …………………………..
Prof. Eke, A.N. Date
External Examiner

DEDICATION

This work is dedicated to GOD ALMIGHTY for HIS Grace and Sustenance throughout the period of my studies in this University, and to the memory of my late mother, Mrs. Ayondu Oraekie . May her gentile soul continue to rest in peace.

ACKNOWLEDGEMENTS

With special gratitude to GOD ALMIGHTY, I appreciate the assistance of my Principal supervisor Dr. V.A Iheagwam whose initiative, understanding and tolerance without which this work would have been very difficult to finish.

My special thanks go to my two other supervisors Dr. Inyama S. C –The former Head of the Department of Mathematics and Computer Science for all his assistance, tolerance, guidance, and understanding from the beginning of my journey to Ph.D till this date and Dr. C.A Nse, whose contributions to the success of this work are immense.

My special gratitude goes to Prof. I. S. Iwueze – the Head of the Department of Mathematics for all his assistance, tolerance, guidance, and understanding ,and Prof. A.M Ette - the Coordinator Postgraduate Studies, Department of Mathematics and Computer Science for all his tolerance and guidance to me.

I must not fail to thank my lovely wife-Barrister Lady Grace Chinyere Oraekie and my kids: Obiora, Nnenna, Amalachukwu, Ego, for supporting me with love and all round encouragement.

I will not also fail to mention the encouragements and assistance I received from Engr. Offurum Julius, and Dr. Ogbonna Chukwudi Justin as well as Dr. Umana R.A.

TABLE OF CONTENTS

ABSTRACT

In this work, necessary and sufficient conditions are investigated and proved for the controllability of nonlinear functional neutral differential equations. The existence, form, and uniqueness of the optimal control of the linear systems are also derived. Global uniform asymptotic stability for nonlinear infinite neutral differential systems are investigated and proved and ultimately, the Shaefers' fixed point theorem is used to forge a new and far-reaching result for the existence of mild solutions of nonlinear neutral differential equations in Banach Spaces.

KEY WORDS:

Relative Controllability, Volterra Integro-Differential Equation, Optimal Control, Complete State, Unsymmetric Fubini Theorem, Neutral Systems, Linearization, Exponential Estimate, , Stability in the Large, mild solution .

CHAPTER 1

GENERAL INTRODUCTION

1.1 Background

Controllability is one of the fundamental concepts in mathematical control theory. It is a qualitative property of dynamical control systems and is of particular importance to the control theorist. In the recent past, the theory of control of deterministic processes with several degrees of freedom appeared to have reached a satisfying stage of completeness. As interpreted by the theory of nonlinear ordinary differential equations, **Iyai (2006)** the fundamental problems of control theory have been mathematically posed and answered and hence the theory has reached a certain degree of stability and perfection. The authors as a result believed that a thorough and careful presentation of the current status of control theory would serve the useful purpose of offering a foundation on which later researches would be based. It is in this intent, that this work: **"Controllability Results for Nonlinear Neutral Functional Differential Systems"** is carried out**.** Our Objective therefore is to present an organized treatment of control theory that could be complete within the limitations set by the restrictions of deterministic problems identifiable in terms of functional differential equations. It is enough to mention here that two kinds of functional differential equations exist.

(a) The Retarded Functional Differential Equation given as

$$\dot{x} = f(t, x_t) \quad ; \quad x(t_0) = \phi = x_{t_0} \qquad (1.\ 1.1)$$

where ϕ is the initial function defined in the delay interval [-h,0], h > 0.

(b) The Neutral Functional Differential Equation given as:

$$\frac{d}{dt}\,[D(t, x_t)] = f(t, x_t) \;;\quad x(t_0) = \phi = x_{t_0} \qquad (1.1.\ 2)$$

where D is a bounded linear operator

It is easily observed that, both equations (1.1.1) and (1.1.2) are characterized by delays. The motivation for this study stems from the fact that most realistic systems should encompass not only the present, but also the past state of the system. This is encountered in many areas of human activities. For a good grasp of the present, (t), some knowledge of the past, (t-h), t ≥ 0, h > 0 , is very important.

In general, differential equations which include the present as well as the past state of any physical system is called a **Delay Differential Equation (or Functional Differential Equations).**

The Retarded Functional Differential Equations **(RFDE) are** characterized by delays in the state of the system. An example is the system

$$\frac{d}{dt}x(t) = x(t-h) \ , \ \ h > 0 \qquad (1.1.3)$$

$On\ the\ other\ hand, Neutral\ Functional\ Differential\ Equations\ (NFDE)$ are those that have delays in the state as well as in the derivatives. An example is the system

$$\frac{d}{dt}\left[x(t) - c\big(x(t-h)\big)\right] \ = \ bx(t-h) \qquad (1.1.4)$$

$$where\ c, b\ , and\ h\ (\ h > 0) are\ constants.$$

Systematic study of controllability started over the years at the beginning of the sixties when the theory of controllability based on the description in the form of state space for both time –varying and time invariant linear control systems was carried out. Roughly speaking, controllability generally means that, it is possible to steer a dynamical control system from an initial state to a final state using the set of admissible controls. Optimal control means doing the same in the best conceivable way. There are many different definitions of controllability which strongly depend on the class of dynamical control systems. In recent years, various controllability problems for different types of nonlinear systems have been considered. However, it should be stressed that, most of the reported work in this direction has been mainly concerned with controllability for linear dimensional systems with constrained control and without delays **(see Klamka (1991), Sun (1996), Underwood and Young (1979)).** Later on delay differential equations came to limelight **(see Nse (2007), Nse (2007)).** A delayed equation on a linear system is one which affects the evolution of the system in an indirect manner.

If we consider the equation

$$\dot{x} \ = \ Ax(t) + Bu(t) \qquad (1.1.5)$$

where A and B are nxn and mxn matrices, we see that the action of the control is direct in that the local behavior of the trajectory is affected only by the local behavior of the control u(t) at time t. It is known that, most natural applications give rise to mechanism of indirect actions where decisions in the control function are shifted, twisted or combined before affecting the evolution thus comprising the delay u(t-h) represented by the system

$$\dot{x} = Ax(t) + Bu(t-h) \qquad (1.1.6)$$

It is well known that the future state of realistic models in the Natural Science, Economics and Engineering depends not only on the present, but also on the past state and at times , even on the derivative of the past state. There are simple examples from Biology (predator-prey, Lodka Volterra, Spread of Epidemics), from Economics (dynamics of capital growth of global economies) and from Engineering (mechanical and aero – space , aircraft stability, automatic steering using minimum fuel and effort, control of high speed, closed air circuit, wind tunnel computer and electric engineering , fluctuations of current in linear and nonlinear circuits, flip-flop circuits and lossless transmission lines).These examples are used to study the stability and time optimal control of Functional Differential Systems. The results of this research effort are, therefore, intended to forge far-reaching solutions to these daily human endeavors.

1.2. Statement of the Problem / Objective.

Our principal objective in this work is to obtain **Necessary and Sufficient Conditions** for controllability, optimal control and stability for Neutral Functional Differential Systems. It is known from **Onwuatu (1993)** that, if a system is relatively controllable, then optimal control is unique and bang-bang. In the light of this, we shall consider the Neutral Volterra Integrodifferential Equation of the form

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t,s)x(s)ds - g(t)\right]$$

$$= A(t)x(t) + \int_0^t G(t,s)x(s)ds + \int_{-h}^0 d_\theta H(t,\theta)u(t+\theta) \qquad (1.1.7)$$

with initial condition $x(t_0) = x_0$, where x$\epsilon E^n$is the state space and u ϵE^m is the control function, H(t ,θ) is an nxm matrix continuous at t and of bounded variation in θ on [-h,0]; h>0 for each tϵ[t_0,t_1] ; t_1 > t_0 . *The nxn matrces* $A(t), C(t,s), G(t,s)$ are continuous in their arguments. The n-vector function g is absolutely continuous.

The above system will be investigated for existence and uniqueness of optimal control by first of all considering the relative controllability of the system.

We shall then forge ahead to achieve solution near the origin to another Neutral Functional Differential Equation of the form

$$\frac{d}{dt}[D(t,x_t)] = L(t,x_t)x_t + f(t,x_t) + \int_{-\infty}^0 A(t)x(t+\theta)d\theta \qquad (1.1.8).$$

where ,

$$L(t,x_t) = \sum_{k=0}^{\infty} A_k x(t-w_k) + \int_{-\infty}^0 A(t,\theta)x(t+\theta)d\theta \qquad (1.1.9)$$

is a bounded linear operator, and f(t , x_t)is a perturbation function. The nxn matrix functions A_k and A(t ,θ) are measurable in (t, s) ϵ ExE, θ ϵ [-∞,0).

The energy method of Alexander Mikhailovick Lyapunov **(1829)** which stipulates that, in a stable system, the total energy in the system will be a minimum at the equilibrium point will be used to establish results. This no doubt will pave the way for discussions on stability of various nonlinear functional equations.

The statements of the problem are thus formulated:

Suppose we are given a Neutral Functional Differential Equation as in equations (1.1.7) and it is required to move the solution $\boldsymbol{x(t)}$ from an initial point $\boldsymbol{x_0}$ at time $\boldsymbol{t_0}$ to a terminal point $\boldsymbol{x_1}$ at time $\boldsymbol{t_1}$.The problem arises as to whether it is possible to carry out this task in finite time. As an illustration, we shall consider the system

$$\dot{x} = -ax(t) \qquad (1.2.0)$$

Clearly, the solution of the above system is

$$x(t) = \mathrm{A}e^{-at} \qquad (1.2.1).$$

If we desire to drive this solution to the origin .that is, null controllability, we observe that, it cannot be achieved in **finite time** because $x(t)$ tends to zero only when t tends to infinity. Since this cannot be achieved in finite time, we need to modify the system to be able to bring $\boldsymbol{x(t)}$ to 0 in **finite time.** The process of modification is called controllability which will answer the controllability problem.

The optimal control problem is formulated as follows: Having guaranteed controllability of the system in question is there an admissible control **u*** such that the solution $\boldsymbol{x(t, \phi, u^*)}$ of the system hits a continuously moving target point in minimum time **t*.** Here u* is the **optimal control** and t* the **minimum time.** Once it is guaranteed that such a control exists, we shall show it is unique and bang-bang.

Finally, we shall ask the question: Is the solution near the origin of system (1.1.8) going to remain quite close for all future times? This is the stability problem which we are desirous to answer in the affirmative.

1.3. Scope of Study.

Differential systems are generally important tools for harnessing different components into a single system and analyzing the inter-relationships that exists between them which otherwise might continue to remain independent of each other. Physical systems which express the present state of solutions are the most common system encountered in the theory of differential equations. The Scope of this work, therefore, is to go beyond these systems and address more realistic systems involving not only the present but also the past states of the system. This is because the latter permeates various aspects of life and has of late triggered interest in research.

Neutral differential equations arise in many areas of applied Mathematics and such equations have received much attention in recent years. For example, the mixed initial boundary hyperbolic partial differential equations which arise in the study of lossless transmission lines can be replaced by an associated neutral differential equation. This equivalence has been the basis of a number of investigations of the stability properties of distributed networks**, (see Iyai (2006), Kwun (1991)).**

It is in this light also that we intend to broaden the scope to involve systems of the Neutral type. This is because in recent years, it has emerged as independent branch of modern research due to its connection to many fields such as continuum mechanics, population dynamics, system theory, biology epidemics and chemical oscillations **(see Balachandran, Balasubramaniam and Dauer (1996), Burton (1983), Corduneanu (1985)).**

1.4. Definitions and Preliminaries

In Mathematical parlance, we consider a system of the form

$$\dot{x} = f(x, t, u) \qquad (1.2.2)$$

where $x \in E^n$ and $u \in E^m$.

This differs from the known familiar one-the usual first order ordinary differential equation **(ODE)** because of the presence of the time function u(t) in the right-hand side of (1.2.2). Many physical processes described by differential equations may have the time-dependence of the process influenced in some manner. This influence is generally referred to as steering or controlling of the process and in (1.2.2) the time function u denotes steering mechanism. Hence, u is often called the steering or control function. Thus, in **Control Theory**, the two dependent variables x and u are called the state variable and the control variable respectively.

In case of air spaceship, the state variable x may refer to the position and velocity of the spaceship, while the control function $u = (u_1, u_2, \ldots, u_m)$ may represent the controllable individual thrusts of the engines.

Suppose now we desire to steer a process (system) from a state x_1 to a state x_2, two very important questions arise:

(i). Does a steering mechanism, a control function u that can be used to steer the process (system) from x_1 to x_2 in a finite time exist? This is the question of **Controllability.**

(ii) Suppose the answer to question **(i)** above is affirmative, among all the possible steering functions that can be used to steer the process (system) from x_1 to x_2 , is there a best one, an optimal one? May be from the point of view of minimizing the travel time, fuel consumption, cost, side effects, or maximizing profit etc? This is the question of **Optimal Controllability.**

A solution of the system (1.2.2) depends on the function u(.) - the notation u (.) is used to denote a function defined on the interval [t_o ,t] . To reflect this dependence, we write a solution to system (1.2.2) as $\psi(x_0, t_0, u(.), t_1)$

The Controllability question (i) may be put in mathematical parlance thus:

Given x_1, t_1 and x_2, does there exist a time $t_2 > t_1$ and a function u (.) such that

$$\psi(x_1, t_1, u(.), t_2) = x_2.$$

If the answer to question (i) is affirmative, the system is said to be controllable, while the system is said to be optimally controllable if the answer to question (ii) is affirmative.

Let us consider the special case in which the right-hand side of system (1.2.2) is linear with constant coefficients. In such a case, we have

$$\dot{x} = A(t)x + Bu(t) \qquad (1.2.3).$$

where A is an nxn-matrix and B is an nxm-matrix. For such linear systems, the concept of complete or null-controllability makes sense.

Definition 1.4.1: (controllability)

The linear system (1.2.3) is said to be controllable if and only if for any initial state x_1 *at time* t_1 there exists steering function u (.) which steers the system from x_1 *at* t_1 to x_2 *at time* t_2 in finite time.

That is for any *initial state* x_1 ,*initial time* t_1 given, there exists *time* t_2 and u (.) such that $\psi(x_1, t_1, u(.), t_2) = x_2$

Definition 1.4.2: (null-controllability)

The linear system (1.2.3) is null-controllable if and only if for any initial state x_1 *at* t_1 there exists steering function u (.) which steers the system from x_1 *at time* t_1 to $x_2 = 0$ *at* t_2 in finite time.

That is for any x_1 ,t_1 given, there exists t_2 and u(.) such that $\psi(x_1, t_1, u(.), t_2) = 0$.

We state, without proof, the following very important result that provides criteria for determining the null-controllability for the system (1.2.3).

Definition 1.4.3: (Reachable Set)

Consider the system (1.2.3) given as

$$\dot{x} = A(t)x + Bu(t) \qquad (1.2.3)$$

Let the solution be $x(t)$ such that

$$x(t) = X(t)X^{-1}(t_0)x_0 + X(t)\int_{t_0}^{t_1} X^{-1}(s)B(s)\,u(s)ds,$$

where $X(t)$ is a fundamental matrix and $X(t_0) = X(0) = I$.

We define the reachable set as

$$R(t_0, t_1) = \left\{\int_{t_0}^{t_1} X^{-1}(s)B(s)\,u(s)ds: u\epsilon U\right\},$$

where U is the set of admissible controls.

Definition 1.4.4: (Attainable Set)

Attainable set is the set of all possible solutions of a given control system. In the case of the system (1.2.3), for instance, it is given as

$$A(t_0, t_1) = \left\{x(t) = X(t)X^{-1}(t_0)x_0 + X(t)\int_{t_0}^{t_1} X^{-1}(s)B(s)\,u(s)ds: u\epsilon U\right\},$$

Evidentl , $R(t_0, t_1)$is a translation of attainable set through the origin x_0, that is

$$\begin{aligned} A(t_0, t_1) &= \{\, x(t) = X(t)X^{-1}(t_0)x_0 + X(t)\int_{t_0}^{t_1} X^{-1}(s)B(s)\,u(s)ds: u\epsilon U \,\} \\ &= X(t)[X^{-1}(t_0)]x_0 + \int_{t_0}^{t_1} X^{-1}(s)B(s)\,u(s)ds: u\epsilon U\,] \\ &= x_0 + \int_{t_0}^{t_1} X^{-1}(s)B(s)\,u(s)ds: u\epsilon U \\ &= x_0 + R(t_0, t_1), \end{aligned}$$

since $X(t)$ is a fundamental matrix and fundamental matrices are invertible .

Definition 1.4.5: (Properness)

The system(1.2.3) given as

$\dot{x} = A(t)x(t) + B(t)u(t)$, is proper on the interval $[t_0, t_1]$ *if* and only i*f* $C^T X^{-1}(t)B(t) = 0, a.e\ on[t_0, t_1]$, implies that $c = 0$.

Here, the set function $g(t) = C^T X^{-1}(t)B(t)$, is called the controllability index.

Lyapunov Function

Consider the system

$$\dot{x} = f(x),\ f(0) = 0 \qquad (1.2.4)$$

where f : D → R^n is continuous, D is a subset of R^n *defined by*

$$D = \{x\epsilon R^n: \| x \| \leq r\}.$$

The solutions of system(1.2.4) are uniquely stable for given initial data t_0 , $|\, t_0| < \infty$and $x\epsilon D$ Here, we shall be concerned with the stability of the trivial solution.

Definition 1.4.6: (positive definite function)

A function V: D→R is said to be positive definite if V varnishes only at the origin and $V(x) > 0\,, for\ all\ x \neq 0$.

Definition 1.4.7: (negative definite function)

A function V: D→R is said to be negative definite if V varnishes only at the origin and $V(x) < 0\,, for\ all\ x \neq 0$.

Definition 1.4.8: (positive semi - definite function)

A function V: D→R is said to be positive semi - definite function if V varnishes only at the origin and $V(x) \geq 0\,, for\ all\ x \neq 0$.

It is negative semi-definite if it varnishes only at the origin and $V(x) \leq 0\ \ , for\ all\ x \neq 0$.

Definition1.4.9: (Lyapunov Function)

Let V: D→R be continuously differentiable and positive definite on D. Let the derivative (called **Eulers' derivative**) of V along the solution path of the system (1.2.4) be defined by

$$\dot{V}(x) = \frac{d}{dx}V(x) = \sum_{i=1}^{n}\frac{\partial V\dot{x}_i}{\partial x_i} = \sum_{i=1}^{n}\frac{\partial V}{\partial x_i}f_i(x),$$

then Vis called a Lyapunov function for the system (1.2.4)

Theorem 1.4.1.

The system (1.2.3) is completely controllable or null-controllable if and only if

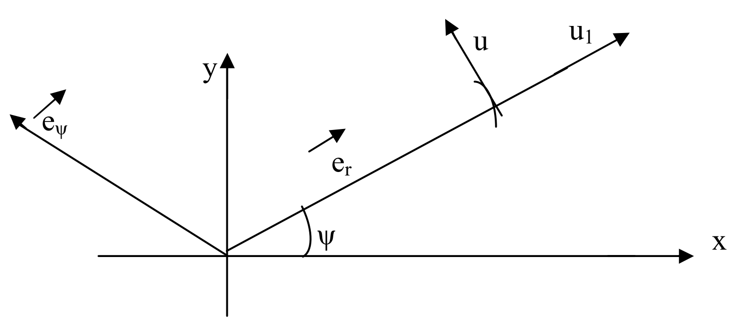
$$Rank(B, AB, A^2B, \dots, A^{n-1}B) = n \qquad (1.2.5)$$

Example 1.4.1

Consider the motion of Satellite in a central gravitational field. Let the kinetic energy T and the potential energy P be given by

$$T = \frac{1}{2}m\,(\dot{r}^2\,, r^2\,\dot{\psi}^2\,),\ P = \frac{ym}{r}$$

where m is the mass of the satellite , if the product of the mass of the planet and the gravitational constant r and ψ are polar coordinate of the satellite (which we idealized as a particle). .



The equations of motion are then

$$\left.\begin{aligned} \ddot{r} &= \frac{-\eta}{r^2} + r\dot{\psi}^2 \\ \ddot{\psi} &= \frac{-2\dot{\psi}\dot{r}}{r} \end{aligned}\right\} \qquad (1.2.6)$$

Particular solution is given by the motion in a circular orbit with ·

$$r = R\,,\ \dot{\psi} = \omega = \sqrt{\eta \frac{1}{R^3}}$$

Deviations from this orbit are to be corrected by two rocket engines with thrust vectors in the direction $\vec{e}_r$ and $\vec{e}_\psi$ respectively. Thus, the equation of motion now becomes

$$\left.\begin{aligned} \ddot{r} &= \frac{-\eta}{r^2} + r\dot{\psi}^2 + u_1 \\ \ddot{\psi} &= \frac{-2\dot{\psi}\dot{r}}{r} + \frac{u_2}{r} \end{aligned}\right\} \qquad (1.2.7)$$

where u_1 and u_2 are the radial and transverse (normal) components of the acceleration respectively (with respect to the circular orbit) as produced by the rocket engines.

Let us now introduce the following new variables:

$$\left.\begin{aligned} x_1 &= r - R \\ x_2 &= \dot{x}_1 = \dot{r} \\ x_3 &= (\psi - \omega t)R \\ x_4 &= \dot{x}_3 = (\dot{\psi} - \omega)R \end{aligned}\right\} \qquad (1.2.8)$$

which represent the (**small**) deviations from the ideal motion

Now, $\dot{x}_2 = \ddot{x}_1 = \ddot{r} = \dot{r}\psi^2 - \frac{\eta}{r^2} + u_1$

$$= (x_1 + R)\,\omega^2 - \frac{\omega^2 R^3}{(x_1+R)^2} + u_1$$

$$= (x_1 + R)\omega^2 - R\,\omega^2\left(1 + \frac{x_1}{R}\right)^{-2} + u_1$$

$$= (x_1 + R)\omega^2 - R\,\omega^2\left[1 - 2\left(\frac{x_1}{R}\right) + 3\left(\frac{x_1}{R}\right)^2 - 4\left(\frac{x_1}{R}\right)^3 + \ldots\right] + u_1$$

$= 3\omega^2 x_1 - 3\omega^2 x_1^2 R^{-1} + 4\omega^2 x_1^3 R^{-2} - 5\omega^2 x_1^4 R^{-3} + \ldots + u_1$ (using Binomial Theorem).

$\Rightarrow \qquad \dot{x}_2 = 3\omega^2 x_1 + u_1 +$ h.o.t (**higher order terms**). (1.2.9)

$$\dot{x}_4 = \ddot{x}_3 = \ddot{\psi} R = (-2\omega\dot{r}\frac{1}{r} + \frac{u_2}{r})R \;= -2\omega R x_2\,(x_1 + R\,)^{-1} + u_2 R(x_1 + R\,)^{-1}$$

$$= \; -2\omega x_2\,(1 + \frac{x_1}{R})^{-1} \;\; + u_2\,(1 + \frac{x_1}{R}\,)^{-1}$$

$$= \; (\,-2\omega x_2 + u_2)\,(1 + \frac{x_1}{R}\,)^{-1}$$

$$= \; (\,-2\omega x_2 + u_2\,)\{1 - (\frac{x_1}{R}\,)^{1} + (\frac{x_1}{R}\,)^{2} - (\frac{x_1}{R}\,)^{3} + \dots\,\}$$

$$= \; -2\omega x_2 + \; u_2 \quad + \text{h.o.t}\,. \qquad (1.3.0)$$

Thus, the linearized system yields

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 3\omega^2\, x_1 \; + u_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -2\omega x_2 + u_2 \end{aligned}\right\} \qquad (1.3.1)$$

with $\bar{x}^T \;=\; (\,x_1\;\; x_2\; x_3\;\; x_4\,)$, and $\bar{u}^T \;= (u_1\; u_2)$

We have the equivalent system (1.2.3) with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

which provides a first approximation to the controlled motion in a neighborhood of the reference orbit.

To determine the controllability of the system, we compute rank (B, AB, A^2B, A^3B) and show that it equals 4.

$$\text{Now, rank (B, AB, A}^2\text{B, A}^3\text{B)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 3\omega^2 & 0 \\ 1 & 0 & 0 & 0 & 3\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2\omega & 0 & 0 & 0 \\ 0 & 1 & -2\omega & 0 & -6\omega^3 & 0 & 0 & 0 \end{pmatrix} = \; 4 = n.$$

Thus, the system is completely controllable. This means that the satellite can be steered to “always” move in the circular orbit.

Suppose one of the rocket engines breaks down or is shut down for some reason; will the system remain completely controllable? Let us suppose the engine providing the radial thrust is shut down, then only the transverse (tangential) thrust is available, the $B^T = (0\ 0\ 0\ 1)$.

Computing, we have

$$\text{Rank}(B, AB, A^2B, A^3B) = \text{rank}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 2 \neq 4$$

The system, therefore, is no longer completely controllable.

Suppose now the engine providing the transverse thrust is shut down, then only the radial thrust is available. Then

$$B^T = (\ 0\ 1\ 0\ 0\ )$$

Computing, we have

$$\text{rank}\ (B, AB, A^2B, A^3B) = \text{rank}\begin{pmatrix} 0 & 1 & 0 & 3\omega^2 \\ 1 & 0 & 3\omega^2 & 0 \\ 0 & 0 & -2\omega & 0 \\ 0 & -2\omega & 0 & -6\omega^3 \end{pmatrix} = 3 \neq 4.$$

The system is again no longer completely controllable. Thus, once one of the engines is shut down, the satellite can no longer be **"constrained"** to move along the circular (reference) orbit.

1.5 .Functional Differential Equations

Definition 1.5.1.

Functional differential equation is an equation (ordinary differential equation) where the derivative at various time instances $t-h_i$, $(i = 0,1,2,\ldots,n)$ is a relation to the state of the equation at equally various time instances. It is a more general type of differential equation. It comprises:

(a) . Retarded equation and (b). Neutral equation.

Definition 1.5.2: (Retarded Function*al* Differential Equation)

Retarded Functional differential equations are differential equations where the derivative of the state is expressed in terms of the state at various time instances $t-h_i$,

$(i = 1, 2, 3,\ldots, n)$

Example 1.5.1

$\dot{x}(t) = 2x(t) + 3x(t-h)$,where $\dot{x}(t)$ is a derivative of a state $x(t)$.

$\dot{x}(t) = 1 + x(t-1) + x(t-2) + x(t)$.

Here, 1 = constant

$(t-1)$= a day or a year ago. (I.e. earlier time)

$(t-2)$= 2 days or 2 years ago.

t = presently or present time.

Difinition1.5.3: (Neutral Functional Differential Equation)

Neutral Functional Differential Equation is the differential equation, where the derivative of the state at previous and present time instances is expressed in terms of time t.

Example 1.5.2

$$\dot{x}(t) - \dot{x}(t-1) = x(t) + 2x(t-1) + 3x(t-2)$$

1.6. Difference between *Retarded* Functional and Neutral Differential Equations

Difference between the ***retarded*** functional differential equation and the **neutral** functional differential equation is that the ***retarded*** functional differential equation contains present time in its derivative, while the **neutral** functional differential equation contains both present and previous (past) time in its derivative. The **retarded** or the **neutral** functional differential equation is called **Delay Equation.**

Usually in the neutral functional differential equation, we define a continuous linear operator **D** to represent the derivative of the state at present, and earlier time instances.

That is, $\mathbf{D}(\boldsymbol{t}, \boldsymbol{x_t})$ is called the **functional difference operator**,

where $(\boldsymbol{t}, \boldsymbol{x_t})$ $\mathrm{D}(t, x_t) = x(t) – x(t-1)$.

We can now rewrite the equation of **Example 1.5.2** in terms functional operator thus:

$$\frac{d}{dt}[D(t, x_t)] = \frac{d}{dt}[x(t) – x(t-1)] = x(t) + 2x(t-1) + 3\,x(t-2) .$$

$$= \dot{x}(t) - \dot{x}(t-1) = x(t) + 2x(t-1) + 3\,x(t-2) .$$

1.7. General Form of Functional Differential Equation.

The initial conditions for general differential equation and functional differential equations are respectively given by:

$$\dot{x} = f(t, x) \;;\; x(t_0) = x_0 \;, \qquad (1.3.2)$$

Here, x_0 is a vector (i.e. initial point is a vector) and

$$\dot{x} = f(t, x_t) \;;\; x(t_0) = \phi. \qquad (1.3.3)$$

Here the initial point φ is a function defined in the delay interval [-h, 0], h > 0 ,

$$x(t_0) = x_{t_0} = \phi$$

Definition 1.7.1: (Retarded functional differential equation)

In Functional Differential Equation of the **retarded type,** whose general equation is given by

$$\dot{x} = f(t, x_t) \;,$$

we prescribe an initial function $x(t_0) = x_{t_0} = \phi$,

over the delay interval [-h, 0], h > 0

Definition1.7.2: (Neutral Functional Differential Equations)

In functional differential equation of the **neutral type**, whose general equation is given by

$$\frac{d}{dt}\left[D(t, x_t)\right] = f(t, x_t),$$

we also prescribe initial function $x(t_0) = x_{t_0} = \phi$, over the delay interval [-h, 0], h > 0

Definition 1.7.3

Let $x(t)$be a function defined over the interval (-∞ ,t) , the function $\boldsymbol{x_t}$ is defined over the delay interval [-h, 0] such that

$$x_t(s) = x(t+s) \; ; s \in [-h, 0].$$

1.8. General Solution Format or Variation of Parameters.

Consider the **delay** equation (**retarded type**)

$$\dot{x} = f(t, x_t) \; ; x(t_0) = x_{t_0} = \phi,$$

The solution format is given by direct integration with respect to time t to get,

$$x(t) = \phi(0) + \int_{-h}^{0} f\big(s, x_t(s)\big) ds \;, \qquad t > 0 \qquad (1.3.4)$$

If it is the **neutral** functional differential equation given by

$$\frac{d}{dt}[D(t, x_t)] = f(t, x_t) \; ; \; x(t_0) = x_{t_o} = \phi,$$

the solution format (variation of parameter) is given as

$$D(t, x_t) = \phi(0) + \int_{-h}^{0} f(s, x_t(s)) \, ds, \qquad t > 0 \qquad (1.3.5)$$

1.9. Volterra Integral Equations

In applied mathematics, mathematical physics, radioactive transfer, theory of population and in engineering, most formulations are often presented in the forms of integral equations.

An integral equation is an equation in which the function to be determined appears under an integral sign.

In ordinary differential equations, integral equations can either be linear or non-linear. The linear integral equations were grouped into two, namely

(a). Fredholm integral equations and (b).Volterra integral equations.

Example 1.9.1 (Fredholm integral equations and/or forms)

$$\int_a^b K(x,y)\,\phi(y)dy = f(x) \qquad \text{(i)}$$

$$\phi(x) - \lambda \int_a^b K(x,y)\,\phi(y)dy = f(x) \qquad \text{(ii)}$$

$$\alpha(x)\phi(x) - \lambda\int_a^b K(x,y)\,\phi(y)dy = f(x) \qquad \text{(iii)}$$

Equation (i) is called **Fredholm integral equation of first kind**. While equation (ii) is the **Fredholm integral equation of the second order** and equation (iii) is the **Fredholm integral equation of the third kind.** In all the three forms of Fredholm integral equations [(i) - (iii)], $K(x,y)$ is called the kernel and $\phi(y)$ which appears under the integral sign is the dependent variable meant to be determined. The integral from a- b may be infinite or may take the forms: (-∞,b] or [a,∞) or (-∞,∞).

Example 1.9.2: (Volterra integral equations and/or forms).

1. $\boldsymbol{f(x) = \int_a^x K(x,y)\,\phi(y)dy}$

2. $f(x) = \phi(x) - \lambda\int_a^x K(x,y)\,\phi(y)dy$

3. $f(x) = \alpha(x)\phi(x) - \lambda\int_a^x K(x,y)\,\phi(y)dy$

We observe that the upper limit in the above equations (1) – (3) is a variable x and not a constant as in the case of Fredholms'.

Definition 1.9.1: (Closed Operators)

An operator T: X → Y , where X, Y are linear spaces is said to be closed if for any sequence $u_n \epsilon D(T)$ such that $u_n \to u$ and $Tu_n \to v$, $u \epsilon D(T)$ and $Tu = v$

1.10. Existence and Uniqueness of Optimal Control for Linear Neutral Volterra Integro-Differential Systems

We recall here that our system of investigation is given by

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t,s)x(s)ds - g(t)\right] = A(t)x(t) + \int_0^t G(t,s)x(s)ds + \int_{-h}^0 d_\theta H(t,\theta)u(t+\theta) \qquad (1.3.6)$$

For purposes of clarity, we define the following terminologies as they relate to system (1.3.6) With solution given by

$$x(t) = X(t,0)\Big[\, x(0) - g(0) \,\Big] + g(t) - \int_0^t \Big(\frac{\partial}{\partial t}\Big) X(t,s)g(s)ds$$

$$+ \int_{-h}^{0} dH_\theta \int_{\theta}^{t+\theta} X(t, s-\theta)H(s-\theta,\theta)u_0(s)ds$$

$$+ \int_0^t \Big[\int_{-h}^{0} X(t, s-\theta)d\theta H^*(s-\theta,\theta)\, u(s)\, ds\Big] \qquad (1.3.7)$$

Definition 1.10.1: (Complete state)

The complete state for system (1.3.6) is given by the set z (t) = {x, u_t } .

Definition 1.10.2: (Relative Controllability)

The system (1.3.6) is said to be relatively controllable on [0,t_1] if for every initial complete state z(0) and $x_1 \in E^n$, there exists a control function u(t) defined on [0,t_1] such that the solution of system (1.3.6) satisfies $x(t_1) = x_1$.

Definition 1.10.3: (Reachable Set)

The reachable set for the system (1.3.6) is given as

$$R(t_1, 0) = \Big\{\int_0^t [\int_{-h}^{0} X(t, s-\theta)d_\theta H^*(s-\theta,\theta)u(s)]ds\Big\} \qquad (1.3.8)$$

Definition 1.10.4: (Attainable Set)

The attainable set for the system (1.3.6) is given as , **A(t, 0)** = $\{x\,(t,\ x_0, u) : u \in U\}$

where , U = $\Big\{\ u \in L_2\big(\ [0, t], E^m\big)\ :\ |u_j| \leq 1\ , j= 1,2,\ldots, m\Big\}$

Definition 1.10.5: (Target Set)

The target set for system (1.3.6) denoted by G (t_1, 0) is given as

G(t_1, 0) = $\{\ x(\ t_1,\ x_0, u) :\ t_1 \geq \tau > t_0$ for fixed τ and $u \in U\}$

Definition 1.10.6: (Controllability Grammian)

The controllability grammian for the system (1.3.6) is given as

$$W(0,t) = \int_0^t z(t,s)z^T(t,s)ds.$$

$where\ \ z(t,s) = \int_{-h}^{0} X(t, s-\theta)d_\theta H^*(s-\theta,\theta)$ and τ denotes matrix transpose.

Definition 1.10.7: (Relative Controllability)

The system (1.3.6) is relatively controllable on [0, t_1] if

$$A(t_1, 0) \cap G(t_1, 0) \neq \emptyset;\ t_1 > 0$$

Definition 1.10.8: (Properness)

The system (1.3.6) is proper in E^n on [0, t_1], if span R(t, 0) = E^n that is if

$c^T[\int_{-h}^{0} X(t, s-\theta)\ d_\theta H^*(s-\ \theta,\theta)] = 0$ (almost everywhere.), $t_1 > 0 \Rightarrow c = 0; c \in E^n$

1.11. Global Uniform Asymptotic Stability for Nonlinear Infinite Neutral Differential Systems

In stability theory, the desire is to achieve that solutions near the origin remain quite close for all future times. The desire to maintain a constant for the solutions of a system over time has given rise to different variants of stability. We have in the literature, uniform stability, asymptotic stability, exponential stability of neutral equations, **(see Cheban (2000), Chukwu (1992), Chukwu (1981), Hale (1977), and Onwuatu (1994)).**

The study of stability of neutral systems has given impetus to the task of investigating the stabilization of nonlinear neutral system. This is the ever growing interest by researchers in stability theory **(see Eke (2000)).**

The methods involved the computation of the eigen-values of certain matrices. With the computer now in vogue, the computations involved are less tedious.

The energy method of **Alexander Mrkhailovick Lyapunov (1829),** which revolves around the notion that in **a stable system, the total energy in the system would be a minimum at the equilibrium point.** The total energy is called the Lyapunov function, whose derivative along the solution path must be negative semi-definite for the solution of the system to have small upper bound, **(see Hmanmed (1986)).** The energy method has provided approval method for discussing stability of various non-linear functional equations. In his work, **Chukwu (1992)** extended the work in **Cruz and Hale (1970)** by monitoring nonlinear functional equations with positive definite Lyapunov functions where explicit solutions cannot be guaranteed. **Hale (1977)** has discussed extensively on the stability of nonlinear neutral equations using various methods, providing exponential estimates of solution of same thereby lending clarity of meaning to exponential stability ,which in **Chukwu (1992)** and **Chukwu (1981)** was extended to asymptotic exponential stability in the large for neutral systems. Stability of perturbations of linear neutral systems has received appreciable emphasis in **Chukwu (1992)** and **Chukwu (1981).**

In **Onwuatu (1993)** stability of infinite neutral systems is reported. We hope to extend the works in **Onwuatu (1993)** to systems of the form.

$$\frac{d}{dt}[D(t,x_t)] \quad = \quad \mathrm{L}(t,x_t)x_t + f(t,x_t) + \int_{-\infty}^{0} A(t)\, x(t+\theta)d_\theta \,, x_{t_0} = \phi. \qquad (1.3.9)$$

(a nonlinear infinite neutral system)

Now, let n be a positive integer and E = (- ∞, ∞) be the real line. Denote by E^n the space of real n- tuples called the Euclidean space with norm denoted by $|\,.\,|$

If J = [a ,b] is any interval of E, L_2 is the Lebesgue space of square integrable functions from J to E^n written in full as $L_2([a, b], E^n)$.

Let h > 0 be a positive real number and let C ([-h,0], E^n) be the Banach space of continuous function with the norm of uniform convergence defined by

$\| \phi \|$ = supϕ(s), -h ≤ s ≤ 0, for $\phi \in C\big([-h, 0], E^n\big)$,

If x is a function from$[-h,\infty)$ *to* E^n, *the* x_t *is a function defined on the onthedela interval given as* $_t(s) = x(t+s); s\epsilon[-h,0], t\epsilon[0,\infty)$.

Consider the nonlinear infinite neutral system.

$$\frac{d}{dt}[D(t,x_t)] = L(t,x_t)x_t + \int_0^\infty A(t)\, x(t+\theta)d_\theta + f(t,x_t) \qquad (1.4.0)$$

where $(t,x_t) = \sum_{k=1}^{\infty} A_k x(t-w_k) + \int_{-\infty}^{0} A(t)\, x(t+\theta)d_\theta$

$L(t,x_t)x_t = \int_{-h}^{0} d_\theta \eta\big(t,s,x(t+s)\big)\, x(t+\theta)$

$$\eta(t,s,\phi,\psi) \geq 0, \text{for } s \geq 0 \text{ , and } \phi,\psi \epsilon C$$

$$\eta(t,s,\phi,\psi) = \eta(t,s,\phi,\psi) \text{ for } t < -h.$$

$\eta(t,s,\phi,\psi)$ is a continuous matrix function of bounded variation in s $\epsilon[-h, 0]$,

$$var\ \eta(t) \leq m(t), m(t)\epsilon L_1.$$

L_1 is the space of integrable functions. Let **Ω** be an open subset of **E**x**C** and D and L be bounded linear operators defined on **E**x**C** into $\mathbf{E^n}$**.**

$|L(t,x_t)x(t)| \leq m(t)\|x_t\|, for\ all\ t\epsilon E, \psi(t) \in$ **C**.

$D(t,x_t) = x(t)g(t,x_t)$, *where*

$g(t,x_t) = \sum_{n=0}^{\infty} A_n(t)\, \phi(t - w_n(t)) + \int_{-h}^{0} A(t,s)\, \phi(s)ds = \int_{-h}^{0} d_\theta H(t,\theta)\, \phi(\theta)$,

where $0 \leq w_n \leq h$ *and* $|\int_{-h}^{0} d_\theta H(t,\theta)\, \phi(\theta)\,| \leq h(\theta)\|\phi\|$

$D(t,x_t)$ *is* ***non − atomic at zero***(differentiable and integrable at zero)

$$\int_{-h}^{0}|A(t,s)|ds + \sum_{n=1}^{\infty}|A_n(t)\,| \leq \delta(\epsilon) \text{ , for all t , where } \delta(\epsilon) \to 0 .$$

f is continuous and satisfies other smoothness conditions.

Consider the system (1.4.1) below:

$$\frac{d}{dt}[D(t,x_t)] = L(t,x_t)x_t + \int_0^\infty A(t,\theta)x(t+\theta)d_\theta + f(t,x_t) \qquad (1.4.1).$$

(Circularity of the function from -∞ to 0, and from 0 to ∞)

We can linearize the system (1.4.1.) as in **Chukwu** (**1992**) by setting $x_t = z$; *a* specified function inside the function $L(t,x_t)x_t$ *to have* $L(t,z)x_t$ *with no loss of generality.*

Thus , the system (1.4.1) becomes

$\frac{d}{dt} D(t, x_t) = L(t, z)x_t + \int_0^\infty A(t, \theta)x(t + \theta)d_\theta + f(t, x_t)$ (1.4.2)

Evidently,

$L(t, z)x_t = \sum_{k=0}^{\infty} A_k x(t - w_k) + \int_{-\infty}^{0} A(t, \theta)x(t + \theta)d_\theta + \int_0^\infty A(t, \theta)x(t + \theta)d_\theta$ (1.4.3)

$L^*(t, z)\ x_t = \sum_{k=0}^{\infty} A_k(t - w_k) + \int_{-\infty}^{\infty} A(t, \theta)x(t + \theta)d_\theta$ (1.4.4)

$The\ representations\ L, L^*$ are the same under the following assumptions

$L(t, z)x_t = \lim_{p\to\infty} \sum_{k=0}^{p} A_k x(t - w_k) + \lim_{M,N\to\infty} \int_M^N A(t, \theta)x(t, \theta)ds$ (1.4.5)

We assume the limits exist, giving finite partial sum for the infinite series and the improper integrals. Thus the system

$L^*(t, z)x_t = \sum_{k=0}^{\infty} A_k(t - w_k) + \int_{-\infty}^{\infty} A(t, \theta)x(t + \theta)d_\theta,$

is finite and well defined function.

In the light of the above,the system (1.4.1) reduces to

$\frac{d}{dt}[D(t, z)x_t] = L(t, z)x_t + f(t, x_t); x(t_0) = \phi \epsilon C$ (1.4.7)

$$where, \quad L(t, z)x_t = \sum_{k=0}^{p} A_k x(t - w_k) + \int_{-h}^{0} A(t, \theta)x(t + \theta)d_\theta,$$

Integrating (1.4.7), after linearizing, we have

$x(t) = x(t, t_0, \phi, 0) + \int_0^t X(t, s)f(s, x_s)ds$, (1.4.8)

Where, X(t ,s) is the fundamental matrix of the homogenous part of the system (1.4.7).

X (t, s) = I (identity matrix); t = s

From the transformation in **Hale (1977),** there is a linear operator T such that

$X_t(s)\phi = (t, s)X(\theta); \theta\epsilon[-h'0].$ (1.4.9)

$X(t + \theta, s) = T(t, s)X(\theta)$

For $\theta = 0$, we have $X(t, s) = T(t, s) = T(t, s)$, where T is defined as follows:

(i) $T(t, s)$ is an operator defined on $C = C([-h, 0], E^n)$, $T(t, s)$ is bounded for T ϵ C.

(ii) $T(0) = I\ and\ T\ is\ strongly\ continuous.$

(iii) $T(t, s)$ is completely continuous in t.

The family $\{ (t, s)\ for\ t > s\}$ is a semi − group of linear transformations,

(**see Chukwu**(**1983**))for these properties. Now writing (1,4.8)in terms of $T(t, s)$,

$we\ have\ ,\ x(t, t_0, \phi, f) = [T(t, t_0)]\phi(0) + \int_{t_0}^{t} X(s, x_s)ds.$ (1.5.0)

We now define the following:

Definition: 1.11.1: (stability)

The trivial solution $x = 0$, of system (1.4.1) is **stable** if for any given $t_0 \epsilon$ E and a positive number $\varepsilon > 0$, *there exists* $\delta = \delta(t_0, \varepsilon)$ such that $\phi \epsilon B(0, \varepsilon)$, implies that

$x_t(t_0, \phi) \epsilon \beta(0, \delta)$ for all $t \geq t0$, $\phi \epsilon C$ and $\mathbf{B}(\mathbf{0}, \mathbf{r})$ is a ball centered at 0, with radius r.

Definition 1.11.2: (uniform stability)

The trivial solution x = 0 of the system (1.4.1) is said to be **uniformly stable** if for any ε >0, there exists $\delta = \delta(\varepsilon)$ (independent of t_0) such that $\phi \epsilon B(0, \varepsilon)$ implies

$x_t(t_0, \phi) \epsilon B(0, \delta)$, for all $t > t_0$

Definition 1.11.3: (asymptotic stability)

The trivial solution $x = 0$, of the system (1.4.1) is asymptotically stable if it is stable such that $\phi \epsilon B(0, \delta)$ implies that $x_t(t_0, \phi) \to 0$, as $t \to \infty$

Definition 1.11.4: (uniform asymptotic stability)

The trivial solution of the system (1.4.1) is uniformly asymptotically stable if the system is uniformly stable and for $\phi \epsilon B(0, \delta)$, implies $x_t(t_0, \phi) \to 0$ as $t \to \infty$.

The solution $x_t(t_0, \phi)$ of system (1.4.1) is exponentially asymptotical stable if there exist constants $k > 0$,and $c > 0$, such that the solution satisfies

$$x_{t_0}(t_0, \phi) = \phi \text{ and } \| x_t(t_0, \phi) \| \leq k e^{c(t-t_0)}$$

Remark: 1.11.1

If the choice of the initial function $\phi \epsilon C([-h,0], E^n)$ is arbitrary, then Definition (1.11.4) is said to be exponentially asymptotically stable in the large and so we have global results that generally defines global conditions.

A homogenous linear neutral equation is given by

$$\frac{d}{dt}\left[D(t, x_t)\right] = L(t, x_t) \qquad (1.5.1.)$$

$D(t, x_t)$ is called the functional difference operator. We now give the condition for the uniform stability of the functional difference operator.

1.12. Existence of Mild Solution of Nonlinear Neutral Differential Equations in Banach Spaces

The primary motivation of the study of neutral functional differential equations is that it has wide range of applications **(see Balachandran and Anandhi (2003), Balachandran and Dauer (1996)). Balachandran and Dauer** (**1996**) have pointed out their application in transmission line theory. They explained that the mixed initial boundary hyperbolic differential equation which arises in the study of lossless transmission lines can be replaced by an associated neutral differential equation. **Asuquo and Usah (2008), Chukwu (1992),** and **Iheagwam and Nse (2007),** have provided complex economic models governed by neutral differential equations and have provided broad policy guidelines for the control of regional economy to equilibrium state.

Neutral systems have also been cited to have applications in population studies and Engineering, in nuclear reactor dynamics, (**see Balachandran and Dauer** (**1996), Fu and Ezzinbi (2003)).**

Many studies have investigated the conditions for the existence of solutions of linear and non-linear neutral systems. Notably, among them are in **Balachandran and Dauer** (**2002), Balachandran and Leelamani (2006), Fu and Ezzinbi (2003).** The presentation in; **Balachandran and Anandhi (2003), Balachandran and Dauer** (**1996**), **Balachandran and Sakthivel (1999)** have introduced a twist in the study of neutral systems by investigating linear and non-linear neutral volterra integro differential equations. These efforts provided more advanced method of integration yielding the variation of parameters **(see Balachandran and Anandhi (2003)).** The theory of neutral differential equation is currently being carried out in Banach spaces, **(see Balachandran and Anandhi (2003), Balachandran and Leelamani (2006), Umana (2008)).**

$$\frac{d}{dt}[x(t)+g(t,x(t),x(u_1(t)),\dots,x(u_m(t)))]$$

$$=L(t,x_t)+h\left(t,x(t),x(v_1(t)),\dots,x(v_n(t))\right)$$

$$x(\theta)=0;\theta\epsilon[-h,0],t\epsilon[0.t_1],t_1>0 \qquad (1.5.2),$$

With the purpose of obtaining mild solutions of the system(1.5.2) in the Banach spaces ,using the **Schaefer's Fixed Point Theorem.**

In the system (1.5.2), g, L, h are the systems parameters. L is the infinitesimal generator of a compact analyti

$L(t,x_t) = \int_{-h}^{0} d\eta(.,s,\phi)\,x(t+s) = \sum_{k=0}^{\infty} A_k(t-w_k) + \int_{-\infty}^{0} A(t,\theta)x(t+\theta)\,d_\theta$

is a bounded linear operator, where the n x n matrix functions A_k, $A(t,\theta)$ are measurable in

((t, s) ϵ E x E, θϵ [- ∞, 0). η is normalized such that

$\eta(t,s,\phi)=0$, $s\geq 0$ for all ϕ

$\eta(t,s,\phi)=\eta(t,-h,\theta)$ for all $s\leq -h$

$\eta(t,s,\phi)$ is continuous from the left in s on (-∞, 0] and has bounded variations on (-∞, 0] for each t, ϕ and there is an integrable function M such that

$\| L(t, x_t) \| \leq$ M(t) $\| x_t \|$,for all t ϵ (-∞,∞), ϕ ϵ (-∞,0].

We assume L(t, ϕ) is continuous. Let 0ϵD(L), then the fractional power L^a for , $0 < a < 1$ as closed linear operator on its domain D (L^a) is dense in X. Furthermore, D(L^a) is a Banach space under the norm

$\| x \|_a = \| L^a x \|$ for all x ϵ D (L^a) and it is denoted by X_a. h is a function defined on the product space JxX^{n+1} into X

g : $[0,t_1]$ x $X^{n+1} \rightarrow X$, is a continuous function.

The delays $u_i(t)$, $v_j(t)$, are continuous scalar valued functions defined on J such that $u_i(t) \leq t$ and $v_j(t) \leq t$.

That is, these are values preceding t. We define the supremum norm on X by

$\| x \| = \max_{x \epsilon X} |x(t)|$.

The imbedding $X_a \rightarrow X_b$ for $0 < b < a < 1$ is compact whenever the resolvent operator L is compact. For semi-group {T(t) }, the following properties will be used:

(i) There is a number $N_1 > 1$ such that $\| T(t) \| \leq N_1$ for all t ϵ [0, t_1].

(ii) For any a > 0, there exists a positive constant N_2 such that

$$\| L^a T(t) \| \leq \frac{N_2}{t^a}, \qquad 0 < t < \tau$$

To study system (1.5.2), we assume the hereditary property of the function.

Let x: (-∞,τ] → X ,x_t is a function defined on the delay interval (-∞ ,0] such that $x_t(\theta) = x(t + \theta)$, belongs to some abstract phase space (-∞, 0] .

In this work, the state space will be the abstract phase space C ([-∞, 0])

Definition 1.12.1: (mild solution)

A function $x(.)$ is called **a mild solution** of the system (1.5.2) if

$$x(t) = 0 \quad , \quad \text{for } t \in (-\infty, 0],$$

the restriction of $x(t)$ to the interval [0, τ] is continuous and for each [0, τ], the function $x(t)$ satisfies system(4.4.3). That is, the function $x(t)$ satisfies the following integral equation:

$$x(t) = T(t)\{0 + g[0, x(u_1(0)), \ldots, x(u_m(0))] - g\{t, x(t), x(u_1(t)), \ldots, x(u_m(t))\}$$
$$- \int_0^t LT(t-s)\, g(s, x(s), x(u_1(s)) \ldots x(u_m(s)))\, ds$$
$$+ \int_0^t T(t-s) h(s, x(s), x(v_1(s)), \ldots, x(v_n(s))) ds \qquad (4.4.3)$$

where $LT(t-s)\, g(s, x(s), x(u_1(s)) \ldots x(u_m(s)))$ is integrable for s ϵ (-∞, t] .

CHAPTER 2.

REVIEW OF THE LITERATURE

2.1 Introduction

As stated earlier, the study of controllability dates back as far as the sixties. Over the years, research has been going on in this area for varying linear, semi- linear and nonlinear systems. There are many definitions of controllability as well as the types which strongly depend on the class of control systems. Various controllability problems for different types of nonlinear differential systems have been considered. However, it should be stressed that, most of the reported works in this area has been mainly concerned with controllability for linear dimensional systems with constrained control and without delays. For instance: Chukwu (**1982)** settled the time optimal control problem of linear neutral systems without delay given by the differential system

$$\frac{d}{dt}[D(t,x_t)] = L(t,x_t) + B(t)u(t) \qquad (2.1.1)$$

$$t \geq 0 \text{ , } x_0 = \phi$$

where the control set is a unit cube in the m-dimensional Euclidean space and the target is a continuous set function in an n-dimensional Euclidean space. In his work, necessary and sufficient conditions for the existence and uniqueness of optimal are given.

Balachandran, Dauer and Balasubramana (**1997)** in their work on the Asymptotic Null Controllability of Nonlinear Neutral Volterra Integrodifferential System, studied the system

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t-s)x(s)ds - g(t)\right] = \mathrm{A}x(t) + \int_0^t G(t-s)x(s)ds + B(t)u(t)$$
$$+ (t, x(t), u(t)) \text{ ; } \qquad (2.1.2)$$
$$x(t_0) = x_0$$

They obtained results using the Leray-Schauder fixed point theorem.

Another important work in the literature without delays in the control is the remarkable work of Sontag (**1994).**He dealt with the questions associated with testing controllability of nonlinear systems both those operating in continuous time, that is, system of the type

$$\dot{x}(t) = f\big(x(t), u(t)\big) \qquad (2.1.3)$$

and discrete time systems described by difference equations

$$x^t(t) = f\big(x(t), u(t)\big) \qquad (2.1.4)$$

where the superscript “t” is used to indicate time shift $x^t(t) = x(t+1)$

In principle he studied controllability from the origin. This is the property that for each state $x(t)\epsilon R^n$ there will be some control driving 0 to x in finite time.

Eke (1990) proved the null controllability of a linear control system by means of the Leray-Schauder's fixed point theorem. The method he used involved the development of sufficient conditions that guarantee the existence of at least one solution of the control system which can be steered to zero in finite time. His system is given as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad \text{, without delays.}$$

Eke (2000) also studied the stabilizability for linear feedback observable systems given by

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(0) &= 0 \\ y(t) &= Hx(t) \end{aligned}\right\} \quad (2.1.5)$$

His paper investigated conditions under which linear control systems which are completely observable are stabilizable.

Another work in the literature without delays is the work of **Eke (1990)** on Total controllability for nonlinear perturbed systems given by

$$\dot{x}(t) = f(t,x) + B(t)u(t) \qquad (2.1.6)$$

$$x(t_0) = x_0$$

and its perturbation

$$\dot{y}(t) = f(t,y) + B(t)u(t) + f(t,y(t),u(t)) \qquad (2.1.7)$$

$$y(t_0) = y_0$$

He used Leray-Schauder fixed point theorem to develop sufficient conditions which guaranteed that whenever an unperturbed nonlinear system is 0-controllable that is, to a target, then so is its perturbation.

Gradually interest began to grow in the study of controllability with delays in state, control or both. Evident in literature are the works of the following authors;

On their work on the relative controllability and Null-controllability of linear delay systems with distributed delays in the state and control, **Iheagwam and Onwuatu (2005)** provided necessary and sufficient conditions for the relative, absolute controllability and null-controllability for the generalized linear delay systems and its discrete prototype given by

$$\dot{x}(t) = L(t,x_t) + \int_{-h}^{0} ds\, H(t,s)u(t+s) \qquad (2.1.8).$$

The paper presents illuminating examples on the previous controllability results by **Olbrot A.W (1972). Klamka .J (2004)** in his research paper on Constrained Controllability of Semi linear Systems with multiple delays in Control tackled systems of the form

$$\dot{x}(t) = Ax(t) + f(x,t) + \sum_{j=0}^{m} B_j\, u(t - h_j) \tag{2.1.9}$$

T$\epsilon[0, T]$, T > 0.

In his work, finite dimensional stationary dynamical control systems described by multiple delays in the control are considered using a generalized open mapping theorem; sufficient conditions for constrained controllability are established.

Onwuatu (1993) also studied delay control systems where he considered the autonomos system

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{p} B_j\, x(t - j) + \sum_{j=0}^{m} C_j\, u(t - j) \tag{2.2.0}$$

t$\epsilon[0,t_1]$, $x(t_0) = \phi(t)$

Here the controllability is assumed to be measurable and bounded on every finite interval. The form of the optimal control is given and the criteria for uniqueness established.

More work on delay control systems can also be found in **Nse (2007)** and **Nse** (**2007)** where he dwelled on the minimum energy of a linear discrete neutral system with delayed control and the constrained controllability of infinite dimensional systems with single point delay in control respectively. On the latter he described the system of the form

$$\dot{x}(t) = Ax(t) + f\big(x(t)\big) + B_0 u(t) + B_1 u(t - h) \tag{2.2.1}$$

for tϵ[0,T], T > h with zero initial conditions $x(0) = 0$, $u(t) = 0$ .For tϵ[-h,0], with x and u taking values in a real Banach space. It should be mentioned that, the above system is semi-linear.

Still on delay controls, **Nse (2007)** exploited necessary and sufficient conditions for the constrained relative controllability of semi linear dynamical neutral systems with multiple delays in state and control given by the system

$$\frac{d}{dt}[x(t - h)] = \sum_{i=0}^{m} A_i x(t - h_i) + f\big(x(t)\big) + \sum_{i=0}^{n} B_i\, u(t - h_i) \tag{2.2.2}$$

It is proved that under suitable assumptions, constrained local relative controllability of linear associate, approximate dynamical systems implies local relative controllability near the origin of the original semi linear abstract dynamical system.

In their study on the controllability of the economic growth of third world countries, **Iheagwam and Nse (2007)** modeled the economic growth of third world countries as a dynamics of growth of capital endowment stock. The dynamics turned out to be a mathematical control problem with delays which is solved to provide broad guidelines in the growth of the economies of the third world countries. Their study recommends that successive recycling of stock to build up capital until the expected target of economic well being is attained. A cartel is needed for the supervision of this growth, from the initial endowment, $x_0,$ to the desired capital stock, x_1. They established controllability results for these dynamics and the desired game strategy for winning, as the dynamics are viewed as differential games of pursuits, remarkably reinforce each other in providing broad policy guidelines for revamping the economy of the third world nations

On the criteria of an attainable set of a Discrete Neutral Control System, **Onwuatu (2002)** dwelled on the system

$$\frac{d}{dt}[x(t) - Ax(t-h)] = \sum_{i=0}^{N} A_i(t-ih) + \sum_{i=0}^{N} B_i u(t-ih) \qquad (2.2.3).$$

His results are investigated in the function space $W_2 \text{x} L_2$ of a linear neutral system with commensurate delays in state and control. Necessary and sufficient algebraic criteria expressed in terms of the system matrices for the closeness are derived.

2.2 Controllability of Ordinary Differential Systems

Controllability of systems with ordinary free part has been a subject of intense research as a result of their wide applications in industry, commerce and engineering.

More so, such systems are easy to handle as tools necessary for their study are now readily available**. (See Asuquo and Usah (2008), Cheban (2000)).** Systems with delay in the control, however pose the obvious challenge of how to handle the lags in the control and provided multiple interests on the subject of controllability. They have diversified current thinking to accommodate the configuration of the complete state z(t) = $\{x(t), u_t\}$ as it is transferred from the initial complete state to final state with the pair $(x(t), u_t)$ changing values simultaneously to open up the area of study known as absolute controllability. **See Iheagwam and Onwuatu (2005)** ,while the spontaneous interest on the transfer of the system at (x_0, u_{t_0}) from initial time t_0, to the state $x(t_1)$ at time t_1 using any control gives impetus to the subject of relative controllability **Iheagwam and Onwuatu (2005)** .**Sebakhy and Bayoumi (1973)** blazed the trail by considering a finite set of first order differential equations of the form:

$$\dot{x}(t) = A(t)x(t) + B(t)\, u(t) + C(t)\, u(t-h) \qquad (2.2.4)$$

where A(t), B(t) and C(t) are nxn, nxm, nxm matrices respectively and h>0 is the delay time. They have obtained a rank condition for the controllability of the system which is

$$\text{rank}\left[\, B, AB, A^2B, \ldots, A^{n-1}B, C, AC, A^2C, \ldots, A^{n-1}C \,\right] = n \qquad (2.2.5).$$

This result has since been extended to systems with multiple delays, (**see Nse (2007)),** research is still scanty on the controllability of the linear systems and its perturbation. It is the investigation of the controllability of the linear systems and its perturbation that was done by **Iheagwam and Onwuatu (2005)** and non linear system using Schauders fixed point theorem.

2.3 Computable Criterion for Controllability.

Success in life revolves around the setting of targets and the achievement of same. Controllability presumes a predetermined target and effort is geared toward the selection of initial points and control energy that will steer the state of system at the initial point to a terminal point (desired) in finite time. The set of such initial points is called the **core of target** denoted by **Co (H),** while the set of terminal points form the **target set**, denoted by **H.**

In any admission policy, the academic requirements for admission describe the case for the award of programme certificate which is the target. Initial grants, initial capital investment, all constitute core of their various targets.

The determination of core of targets and their properties are important to administrators, business managers, public and civil servants, and government in realizing their objective with minimum waste. The advantage perhaps underscores the growing importance in recent times of the subject of cores of targets and controllability of differential systems.

Efforts have been made by **Hajek (1974)** and **Chukwu (1986)** to investigate the compactness of core of targets for linear and non linear ordinary control systems respectively.

Hajek (1974) exploited the analyticity of the fundamental matrix of the homogenous differential system

$$\dot{x} = Ax \qquad (2.2.6)$$

and buttressed by a weak compactness argument was able to establish the closedness of core of targets for the ordinary control system

$$\dot{x} = Ax(t) + Bu(t) \qquad (2.2.7)$$

where A and B are constant matrices. Employing arguments from convex set theory Hajek not only proved the convexity of cores of targets but established their boundedness, hence resolving the problem of compactness of cores for system (2.3). Chukwu carried the result of

Hajek to nonlinear systems. **Chukwu (1987)** in his paper extended with same modifications the results of Hajek, to delay systems. He used the concept of asymptotic direction and Hajek-like arguments to establish that a strong relationship exists between the compactness of cores of targets for a linear delay system and controllability of a related system. **Iheagwam (2002)** most recently has extended most results in **Hajek (1974)** and **Chukwu (1987)** to systems with distributed delays in control, establishing among others a strong relationship between cores of targets and the relative controllability of the system of interest.

2.4 Controllability of Non Linear Systems

Several authors have studied the controllability of non-linear systems with the aid of Schauders fixed point theorem, in recent years. (**See Chukwu (1986), Chukwu (1992)), Klamka (1976) and Wei (1976)**. Especially in the paper by **Klamka (1976)** the problem of global relative controllability of certain class of non linear systems with distributed delays in control has been considered.

Jerzy Klamka (1980) extended the results of the papers **Wei (1976)** by considering a more general class of nonlinear time-varying systems with distributed delays in control, using Schauder's fixed point theorem, sufficient conditions for global and local relative controllability are defined.

2.5 Relative Controllability of Non linear Neutral Systems

The primary motivation for the study of neutral integrodifferential equations is the application to transmission-line theory **(see Balachandran and Dauer (1996))** .It is known that the mixed initial-boundary hyperbolic partial differential equation which arises in the study of lossless transmission lines can be replaced by an associated neutral differential equation. The equivalence has been the bases of a number of investigations of the stability properties of distributed network **(see Conner (1978)) .**In particular, models for systems with delay in control occur in population studies and in some complex economic systems. More specifically, models for systems with distributed delays in the control occur in the study of agricultural economic and population dynamics.

Volterra Integrodifferential equations occur in applied Mathematics **(see Burton (1983)).** In **Gyori and Wu (1991)** a simplified model for compartmental systems with pipes is represented by nonlinear neutral volterra integrodifferential equation. The problem of controllability of linear neutral systems has been investigated by several authors**-(see Balachandran and Sakthivel (1999), Fu (2003), Chukwu (1982), Balachandran and**

Leelamani (2006), Underwood and Chukwu(1988) studied the null controllability for such systems.

Further, **Chukwu (1987)** considered the Euclidean controllability of a neutral system with nonlinear base.

Onwuatu (1984) discussed the problem for nonlinear systems of neutral functional differential equations with limited controls. **Gahl (1978)** derived a set of sufficient conditions for the controllability of nonlinear neutral systems through the fixed point method. **Balachandran and Dauer (1989)** investigated the relative controllability of nonlinear systems with distributed delays in control. In their paper **(see Balachandran and Dauer (1996)),** Balanchandran and Dauer derived sufficient conditions for the relative controllability of nonlinear neutral Volterra Integro-differential systems with distributed delays in the control variables. The results were obtained by using Schauder's fixed point theorem.

2.6 Null Controllability of Functional Differential Systems.

The concept of controllability plays a major role in finite-dimensional control theory, so it is natural to try to generalize this to *INFINITE-dimensional systems*. **Balachandran and Leelamani (2006),** Controllability is the property of being able to steer between two arbitrary points in the state space. For continuous time-invariant linear systems in finite-dimensional space, the concepts of controllability and reachability have been studied in the literature. A weaker condition than exact controllability is the property of being able to steer all points to the origin. This has important connections with the concept of stabilizability. Several authors have studied the null controllability of various kinds of dynamical systems. **(see Chukwu (2001)).** Neutral differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. The theory of functional differential equations with unbounded delay has been studied by several authors **(see Balachandran and Anandhi (2003)).** Almost all the work deals with the Cauchy problem,

$$\dot{x}(t) = f(t, x_t), \quad t > \delta, \; x_0 = \phi \qquad (2.2.8)$$

where x_t represents the "history" of x at t, the values x(t) belong to same finite-dimensional space, and f is a function, usually continuous on appropriate spaces. Nevertheless, this class of equations does not include partial integrodifferential equations with infinite delay, which arise, for example, in the study of conduction in materials with memory of population dynamics for spatially distributed populations. Besides, it is well known that the behavior of the first and second order abstract Cauchy problems is different in many aspects. For these reasons, there has been an increasing interest in studying equations that can be described in the form:

$$\dot{x}(t) = Ax(t) + f(t, x_t), \; t \geq \delta \qquad (2.2.9)$$

where A is the infinitesimal generator of strongly continuous semi-group of linear operators on a Banach space X. We call these equations as abstract retarded functional differential equations.

Similarly, there exists an extensive theory for ordinary neutral functional differential equations, which includes qualitative behavior of classes of such equations and applications to biological and engineering processes. **Balachandran and Leelamani (2006)** in their paper, studied the controllability of the equations that can be modeled in the form:

$$\frac{d}{dt}\left[\, x(t) + f(t, x_t) \,\right] = A\, x(t) + G(t, x_t) \qquad (2.3.0)$$

where the initial condition x_δ and f and G are appropriate functions. These functions will be called abstract neutral functional differential equations with unbounded delay. As a motivation example for this class of equations they considered the boundary value problem.

$$\frac{\partial}{\partial t}\left[u(t, \xi) + \int_{-\infty}^{t}\int_{0}^{\pi} b\,(s - t, \eta, \xi)\,u(s, \eta)ds\right] = \frac{\partial^2}{\partial \xi}u(t, \xi) + a_0(\xi)u(t, \xi) + a_1(t, \xi)$$

$$+ \int_{-\infty}^{t} a(s - t)u(s, \xi)ds\,;$$

$t \geq 0 \quad , \quad \theta \leq \xi \leq \pi.$

$u(t, 0) = u(t, \pi) = 0, \quad t \geq 0$

$u(\theta, \xi) = \Phi(\theta, \xi), \quad \theta \leq 0, \quad 0 \leq \xi \leq \pi \qquad (2.3.1)$

where the functions a_0, a, a_1, b , and ϕ satisfy appropriate conditions.

These problems arise from control systems described by abstract retarded functional differential equations with a feedback control governed by a proportional integro-differential system **(see Ananjevskii and Kolmanovskii (1990)).** On the other hand, some abstract retarded functional differential equations can be conveniently transformed into abstract neutral functional differential equations. Consider the equation or the system (2.2.9) with

$$F(\psi) = \int_{-\infty}^{0} C(\theta)\psi(\theta)d\theta \qquad (2.3.2)$$

where C is a strongly continuous map of continuous operators from X into X.

Assume that we can decompose $C(s) = L(s) + N(s)$

where L and N are also strongly continuous maps of continuous operators and further the L(s) are linear. We define the operator V(t) by

$$V(t)x = \int_0^t L(s)xds \qquad (2.3.3)$$

Then systems (2.2.9) can be transformed into an abstract neutral functional differential equation

$$\frac{d}{dt}\left[x(t) + \int_{-\infty}^{t} V(t-s)x(s)ds\right] = Ax(t) + \int_{-\infty}^{t} N(t-s)x(s)ds \qquad (2.3.4)$$

Which has the form (2.3.0) and in some cases depending on V and N, it is easier to treat than the original equation. Motivation for neutral functional differential equations can be found in **Balachandran and Dauer (1996), Chukwu (2001)**.

There are several papers which have appeared on the controllability of non linear systems in infinite-dimensional spaces. **Balachandran and Anandhi (2003)** discussed the controllability of neutral functional integro-differential systems in abstract phase space, with the help of Schauder's fixed point theorem.

Recently, **Fu and Ezzinbi (2003), Fu (2004)** studied the same problem in abstract phase space for neutral functional differential systems and nonlinear neutral systems with unbounded delay by utilizing the Sadovskii fixed point theorem. **Balachandran and Leelamani (2006)** in their paper studied the null controllability of neutral evolution integrodifferential systems with infinite delay in utilizing the technique of **Fu (2004)**. The results are generalizations of the results established by **Fu (2003, 2004).**

We shall ,therefore, forge ahead in this work to employ modern techniques of approach to obtain results and ultimately employ some fixed point theorems to obtain “Mild” solutions of Nonlinear Neutral Differential equations in Banach Spaces.

CHAPTER 3:

METHODOLOGY

3.1. Introduction

The pioneering work of Vito Volterra on the Integration of the differential equations of dynamics and partial differential dynamical systems published in **1884** gave vent to the conception of integral equation of volterra type **(see Robertson and Oconnor (2005)).** It is equally observed in **Balachandran and Dauer (1989)** that the mixed initial boundary hyperbolic partial differential equation which arises in the study of lossless transmission lines can be replaced by an associated neutral differential equation. This equivalence has been the basis of a number of investigations of the stability properties of distributed network (see **Balachandran, Dauer and Balasubramaniam (1997)),** which study has been extended to compartmental models governed by neutral volterra integro-differential equations. Compartmental models have been found in **Burton (1983)** to have numerous applications in Applied Mathematics; these models are used to vividly describe the evolutions of systems, in theoretical epidemiology, physiology, population dynamics chemical reaction kinetics and the analysis of ecosystems **(see Gyori and Wu (1991)).** Most of these models can be divided into separate compartments. A paradigm for such systems can be seen as one in which compartments are connected by pipes through which materials pass in definite time. An example of compartmental model is given in **Gyori and Wu (1991)** as the radio cardiogram where the two compartments correspond to the left and right ventricles of the heart and the pipe between these compartments represent the pulmonary and systemic circulations. Other applications of volterra integro-differential equation arise in tracer kinetics in the modeling of uptake of potassium by red blood cells as well as in modeling the kinetic of lead in a body **(see Burton (1983)**, **Gyori and Wu (1991))**. The wide application of volterra integro-differential equations in Bio-Mathematics and economic models underscores the immense interest the study has generated. Literature on the relative controllability of volterra equations is still scanty. However, sufficient conditions for the relative controllability of non-linear neutral volterra integro-differential equations have been provided in **Balachandran and Dauer (1989)**. However, the systems with delays in the state, investigation into their relative controllability are still attracting attention and interest. Optimality conditions for the relative controllability of neutral volterra integro-differential equation, is yet to be reported; though there are studies in the optimal controllability of ordinary and functional differential systems.

From these studies, **Chukwu (1988), Hmanmed (1986), Klamka (1976),** we gain clarity of meaning and understanding of the conceptual frame work of optimal controllability.

In this work, we shall consider the neutral Volterra integro-differential equation of the form

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t,s)x(s)ds - g(t)\right] = A(t)x(t) + \int_0^t G(t,s)x(s)ds + \int_{-h}^{0} d_\theta H(t,\theta)u(t+\theta) \qquad (3.1.1)$$

with the main objective of investigating the existence and uniqueness of the optimal control for the system (3.1.1).

3.2. Description of System.

Consider the neutral Volterra integro-differential system with distributed delays in the control given by system (3.1) with the initial condition $x(0) = x_0$ $.Here, x\epsilon E^n$ and u is an admissible square integrable m-dimensional vector function; with $|u_j| \leq 1$, j = 1,2,0 ..., m. H(t, θ) is an nxm matrix continuous at t and of bounded variation in θ on [-h, 0]; h > 0 for each $t\epsilon[0,t_1], t_1 > 0$. *The nxn matrices* $A(t), C(t,s), G(t,s)$ are continuous in their arguments. The n-vector function g is absolutely continuous.

The integral is in the Lebesgue-stieltjes sense and is denoted by *the symbol* d_θ.

In this work, the state space denoted by Q is the Banach space of continuous $E^n xE^m$ valued functions defined on $[0,t_1]: t_1 > 0$ with the norm

$$\|(x,u)\| = \|x\| + \|u\| \quad ,$$

where $||x|| = \sup\{ \,|x(t)| : t\epsilon[0,t_1]\}$ *and* $\|u\| = \sup\{\,|u(t)| : t\,\epsilon[0,t_1]\}$.

That is $Q = E^n[0,t_1]xE^m[0,t_1]$ *where* $C^n[0,t_1]$ is the Banach space of continuous E^n-valued functions defined on *on* $[0,t_1]$*with the supremum norm.*

The control space will be the Lebesgue space of square integrable functions, $L_2([0,t_1],E^m)$, where m is a positive integer. The constraint control set U is closed and bounded subset of L_2.

Let, h > 0, be given. *For a function* $u: [-h,t_0] \to E^m$ *and* $t\epsilon[0,t_1]$,

we use the symbol u_t to denote the function defined on the delay interval [-h ,0] by

$$u_t(s) = u(t+s)\,, for \;\; s\epsilon[-h,0].$$

3.3 .Variation of Constant Formula.

By integrating the system (3.1.1), we obtain an expression for the solution as in **Balachandran, K. and Dauer, J.P (1989).**

$$x(t) = X(t,0)[x(0) - g(0) + g(t)$$

$$-\int_0^t \left[\frac{\partial}{\partial t}\right] X(t,s)g(s)ds + \int_0^t X(t,s)[\int_{-h}^0 d_\theta H(s,\theta)u(t+s)]ds \quad (3.1.2)$$

where x(0) is the state vector at t = 0.

$X(t,s)$ *and* $\left[\frac{\partial}{\partial t}\right] X(t,s)$ *are continuous matrices satisfying*

$$[\ \left[\frac{\partial}{\partial t}\right] X(t,s) - \int_0^t \left[\ \frac{\partial}{\partial t}\ \right] X(t,r)C(r,s)dr + C(t,s) - X(t,s)A(t) - \int_0^t X(t,r)\, G(r,s)dr$$

where X(t ,t) = I (the identity matrix). That is, X(t, s) is the fundamental matrix for the homogenous part of the system (3.1.1).

A careful observation of the solution of the system (3.1.1) given as (3.1.2) shows that the values of the control u(t) for t ϵ[-h, t_0] enter the definition of the complete state thereby creating the need for an explicit variation of constant formula. The control in the last term of the formula (3.1.2), therefore, has to be separated in the intervals [-h, 0] and $[0, t_1]$. To achieve this, that term has to be transformed by applying the method of Klamka in **Klamka (1980)** Finally, we interchange the order of integration using the **unsymmetric Fubini's theorem** to have.

$$x(t) = X(t,0)[x(0) - g(0) + g(t)$$
$$-\int_0^t \left[\frac{\partial}{\partial t}\right] X(t,s)g(s)ds + \int_0^t [\int_{-h}^0 d_\theta H(s,\theta) \int_{0+\theta}^{t+\theta} X(t,s)H(s-\theta,\theta)\, u(s)]ds \quad (3.1.3)$$

Simplifying (3.1.3), we have

$$x(t) = X(t,0)[\ x(0) - g(0)\ ] + g(t) - \int_0^t \left(\frac{\partial}{\partial t}\right) X(t,s)g(s)ds$$
$$+ \int_{-h}^0 dH_\theta \int_\theta^{t+\theta} X(t, s-\theta)H(s-\theta,\theta)u_0(s)ds$$
$$+ \int_{-h}^0 dH_\theta \int_\theta^{t+\theta} X(t, s-\theta)H(s-\theta,\theta)u(s)ds \quad (3.1.4).$$

Using again the unsymmetric Fubini's theorem on the change of order of integration and incorporating H* as defined below

$$H^*(s,\theta) = \begin{cases} H(s,\theta) \text{ for } s \le t \\ 0 \quad \text{for } s \ge t \end{cases} \quad (3.1.5)$$

The formula (3.1.4) becomes

$$x(t) = X(t,0)\left[\ x(0) - g(0)\ \right] + g(t) - \int_0^t \left(\frac{\partial}{\partial t}\right) X(t,s)g(s)ds\ +$$
$$\int_{-h}^0 dH_\theta \int_\theta^{t+\theta} X(t, s-\theta)H(s-\theta,\theta)u_0(s)ds$$
$$+ \int_0^t \left[\int_{-h}^0 X(t, s-\theta)d\theta H * (s-\theta,\theta)\ u(s)\ ds\right] \quad (3.1.6)$$

Integration is still in the Lebesgue Stieltjes sense in the variable θ in H

For brevity, let

$$\beta(t) = X(t,0)[x(0) - g(0)] + g(t) - \int_0^t (\frac{\partial}{\partial t}) X(t,s)g(s)ds \qquad (3.1.7).$$

$$\mu(t) = \int_{-h}^0 dH_\theta \int_\theta^{t+\theta} X(t, s-\theta)H(s-\theta,\theta)u_0(s)ds \qquad (3.1.8)$$

$$z(t,s) = \int_{-h}^0 X(t, s-\theta)d\theta H * (s-\theta,\theta) \qquad (3.1.9).$$

Substituting (3.1.7), (3.1.8), and (3.1.9) in (3.1.6), we have a precise variation of formula for the system (3.1.1) as

$$x(t,x_0,u) = \beta(t) + \mu(t) + \int_0^t z(t,s)u(s)ds \qquad (3.2.0)$$

3.4.THE FOLLOWING METHODS/TECHNIQUES WERE USED TO OBTAIN RESULTS:

3.4.1.Relative Controllability Technique.

It is known from **Onwuatu (1993)** that, if the system relatively controllable, then optimal control is unique and bang-bang. In the light of this, we shall consider the Neutral Volterra Integrodifferential Equation of the form

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t,s)x(s)ds - g(t)\right] = A(t)x(t)$$

$$+ \int_0^t G(t,s)x(s)ds + \int_{-h}^0 d_\theta H(t,\theta)u(t+\theta) \qquad (1.1.7)$$

with initial condition $x(t_0) = x_0$, where $x \epsilon E^n$ is the state space and $u \epsilon E^m$ is the control function, H(t ,θ) is an nxm matrix continuous at t and of bounded variation in θ on [-h,0]; h>0 for each $t \epsilon [t_0, t_1]$; $t_1 > t_0$ .The nxn matrices $A(t), C(t,s), G(t,s)$ are continuous in their arguments. The n-vector function g is absolutely continuous.

The above system will be investigated for existence and uniqueness of optimal control by first of all considering the relative controllability of the system.

3.4.2.TheEnergy Method of Alexander Mikhailovick Lyapunov(1829)

We shall then forge ahead to achieve solution near the origin to another Neutral Functional Differential Equation of the form

$$\frac{d}{dt}[D(t,x_t)] = L(t,x_t)x_t + f(t,x_t) + \int_{-\infty}^0 A(t)x(t+\theta)d\theta \qquad (1.1.8).$$

$$where\ , L(t,x_t) = \sum_{k=0}^{\infty} A_k x(t-w_k) + \int_{-\infty}^0 A(t,\theta)x(t+\theta)d\theta \qquad (1.1.9)$$

is a bounded linear operator, and $f(t, x_t)$ is a perturbation function. The nxn matrix functions A_k and A(t ,θ) are measurable in (t, s) ϵ ExE, θ ϵ [-∞,0).

The energy method of Alexander Mikhailovick Lyapunov **(1829)** which stipulates that, in a stable system, the total energy in the system will be a minimum at the equilibrium point will be used to establish results.

3.5. Basic Set Functions and Properties

Applications will be made of the following Basic Set Functions and Properties upon which our study hinges: Reachable Set, Attainable Set, Target Set , Controllability Grammian and Controllability Index of the System (3.1.1).

3.6. Controllability Conditions or Controllability Standard

Applications will be made of the following controllability conditions to establish results:

1. The non-emptiness of the intersection of two set functions- attainable set $\boldsymbol{A(t_0, t_1)}$, and target set $\boldsymbol{G(t_0, t_1)}$ is equivalent to the controllability of the system.

$That\ is, A(t_0, t_1) \cap G(t_0, t_1) \neq \phi,$ implies that the system is controllable.

2. The controllability Grammian or Map ***or map*** $W(t_0, t_1)$ is invertible and the invertibility of the grammian guarantees the controllability of the system. The invertibility of the grammian means that the rank of the grammian must be equal to n.

$That\ is\ , W(t_0, t_1) = rank \int_{t_0}^{t_1} [X^{-1}(s)B(s)][X^{-1}(s)B(s)]^T ds = n, for\ t\epsilon[t_0, t_1], t_1 > t_0,$

3. R.E. Kalmans' Controllability Criterion (theorem) **.**Consider the system given by

$$\dot{x}(t) = Ax(t) + Bu(t) \quad ,$$

$where\ A, B\ are\ constant\ nxn, nxm\ matrices.$Then, the system is proper if and only if

$$rank[B, AB, A^2B, \dots, A^{n-1}B] = n.$$

4. $0\epsilon Interior\ R(t_0, t_1),$ Implies that the system is controllable.

5. For the time varying systems and dynamical systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

is proper if the controllable index z(t) is zero such that c = 0

$$that\ is\ , z(t) = c^T X^{-1}(t)B(t) = 0, \Rightarrow c = 0$$

3.6.1. OPTIMALITY CONTROL

Theorem 3.6.1. **(Existence of Optimal Control)**

Let g(t) be a continuous target function, and consider the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t).$$

If there is an admissible control $u\epsilon C^m\ and\ a\ time\ t_1 \geq 0\ , for\ \ x(t_1, u) = g(t_1),$ then, there is an optimal control.

Theorem3.6.2 (Necessary Condition for Optimality)

$If\ u^*$is an optimal control with t^* the minimum time, then u^* is of the form

$u^*(t) = sgn[c^T X^{-1}(t)B(t)] on [0, t] for\ some c \neq 0.$

3.7 . Bang-Bang Principle

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$Let\ U = \{u \epsilon E^n : \|u_j\| \leq 1\} \qquad \text{(a)}$$

be a set of admissible controls with limited energy and,

$$Let\ U^0 = \{\ u^0 \epsilon E^n : \|\ u_j{}^0\ \| = 1\ \} \qquad \text{(b)}$$

be a set of admissible controls with unlimited energy **(Bang-Bang Control).**

Using the control set (a), we have the reachable set as

$$R(t_0, t_1) = \{\int_{t_0}^{t_1} X^{-1}(s)\, B(s)u(s)ds : u \epsilon U\ \}.$$

If we use the second control set (b), we have the reachable set as

$$R^{\mathbf{0}}(t_0, t_1) = \{\int_{t_0}^{t_1} X^{-1}(s)\, B(s)u^0(s)ds : u^0 \epsilon U^0\ \}$$

The Bang- Bang Principle simple states that $R(t_0, t_1) = R^0(t_0, t_1)\ for\ each\ t.$

$$R(t_0, t_1) = R^0(t_0, t_1)\ for\ each\ t.$$

The interpretation is that any point that can be reached by an admissible control u at any time t can be reached by a bang-bang control u^0 *at the same time*.

3.8. Control Theorem on Nonlinear Systems

Theorem 3.8.1

Consider the system, $\dot{x}(t) = Ax(t) + Bu(t) + f(t, x, u)$

where , $Ax(t) + Bu(t)$ is the linear part and $f(t, x, u)$is the perturbation function.

If the linear part is controllable and the perturbation is bounded, then the nonlinear system is controllable.

3.9. Lyapunov Stability Theorems for Autonomous Systems

Consider the system

$$\dot{x} = f(x)\ ,\ f(0) = 0 \qquad (1.2.4)$$

where $f: D \rightarrow R^n$ *is continuous*, D *is a subset of* R^n defined by

$$D = \{x \epsilon R^n : \|\ x\ \| \leq r\}.$$

The solutions of the system (1.2.4) are uniquely stable for given initial

$data\ t_0, |t_0| < \infty\ and\ x\epsilon D.$

Here, we shall be concerned with the stability of the trivial solution.

Theorem 3.9.1. (Lyapunov Stability Results)

Let V: D→R be a Lyapunov function for the system(1.2.4) on D. Then:

(i) $(i)\ If\ \ \dot{V}(x)$ is negative semi-definite, then the trivial solution of the system (1.2.4) is stable.

(ii) $(ii).\ If\ \ \dot{V}(x)$ is negative definite, then the trivial solution of the system (1.2.4) is asymptotically stable.

(iii) if $\dot{V}(x)$ is positive definite, then the trivial solution of the system (1.2.4) is unstable**.**

3.10. Existence Theorems

Theorem 3.10.1. (Schauder's Fixed Point Theorem)

If A is a closed, bounded and convex subset of a Banach space B and if the map T: A → A is completely continuous, then there is a point zϵA such that T(z) = z, that is ; z is a fixed point

Theorem 3.10.2 (Schaefer's fixed point theorem),

Let B be a normed linear space. Let g: B → B be completely continuous operator, that is, it is continuous and image of any bounded subset is contained in a compact set, and let **Qg** = {x ϵ B: x = λgx ,for λ ϵ (0, 1), that is, 0 < λ < 1},then either Qg is unbounded or g has a fixed point.

CHAPTER 4

MAIN RESULTS

Preamble.

In this chapter, we consider the main results.

We recall here that our system of investigation is given by

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t,s)x(s)ds - g(t)\right] = A(t)x(t) + \int_0^t G(t,s)x(s)ds$$
$$+ \int_{-h}^0 d_\theta H(t,\theta)u(t+\theta) \qquad (4.1.1)$$

with initial condition $x(t_0) = x_0$,where xϵE^n is the state space and u ϵE^m is the control function, H(t ,θ) is an nxm matrix continuous at t and of bounded variation in θ on [-h,0]; h>0 for each tϵ[t_0,t_1] ; $t_1 > t_0$ .The nxn matrices $A(t)$,$C(t,s)$,$G(t,s)$ are continuous in their arguments. The n-vector function g is absolutely continuous.

4.1. Relative Controllability Results

We now state and prove the following theorems that guarantee relative controllability of the system under study

Theorem 4.1 (Relative controllability Result)

Consider the system

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t,s)x(s)ds - g(t)\right] = A(t)x(t) + \int_0^t G(t,s)x(s)ds$$
$$+ \int_{-h}^0 d_\theta H(t,\theta)u(t+\theta) \qquad (4.1.1)$$

with the same conditions on the systems parameters as in (4.1.1), then the following statements are equivalent*:*

(1) The system (4.1.1) is relatively controllable on [0 , t_1]

(2) The controllability grammian W (0, t_1) of system (4.1.1) is non-singular.

(3) The system (4.1.1) is proper on [0, t_1].

Proof.

Straight forward from the arguments in; Ananjevskii and Kolmanovskii (1990), Angell (1990), Balachandran and Dauer (2002).

Theorem 4.2 (Relative **controllability Result)**

The system (4.1.1) with its standing hypothesis is relatively controllable if and only if 0 ϵ Int **R (** $\boldsymbol{t_1}$ **, 0)** for $t_1 > 0$.

Proof

The reachable set **R(** t_1**,0)** is a closed and convex subset **(compact subset)** of E^n. Therefore, a point $z_1 \epsilon$ E^n on the boundary implies there is a support plane $\boldsymbol{\pi}$ of **R(** $\boldsymbol{t_1}$**, 0)** through z_1.

That is, $c^T(z - z_1) \leq 0$ for each z ϵ **R (** $\boldsymbol{t_1}$ **, 0)**, where c ≠ 0 is an outward normal to the support plane $\boldsymbol{\pi}$.

If u_1 is the control corresponding to z_1, we have

$$c^T \int_0^{t_1} \left[\int_{-h}^0 X(t_1, s-\theta) d_\theta H^*(s-\theta,\theta)\right] u(s) ds$$

$$\leq c^T \int_0^{t_1} \left[\int_{-h}^0 X(t_1, s-\theta) d_\theta H^*(s-\theta,\theta)\right] u_1(s) ds \qquad (4.1.2)$$

for each u ϵ U. Since U is a unit sphere the inequality (4.1.2) becomes

$$\left| c^T \int_0^{t_1} \left[\int_{-h}^0 X(t_1, s-\theta) d_\theta H^*(s-\theta,\theta)\right] u(s) ds \right|$$

$$\leq \left| c^T \int_0^{t_1} \left[\int_{-h}^0 X(t_1, s-\theta) d_\theta H^*(s-\theta,\theta).1\right] \right| ds$$

$$= c^T \int_0^{t_1} \left[\int_{-h}^0 X(t_1, s-\theta) d_\theta H^*(s-\theta,\theta).1\right] \operatorname{sgn} c^T \left[\int_{-h}^0 X(t_1, s-\theta) d_\theta H^*(s-\theta,\theta) ds\right] \qquad (4.1.3)$$

Comparing (4.1.2) with (4.1.3), we have

$$u_1(t) = \operatorname{sign} c^T \left[\int_{-h}^0 X(t_1, s-\theta) d_\theta H^*(s-\theta,\theta)\right]$$

More so, as z_1 is on the boundary since we always have 0ϵ **R(** $\boldsymbol{t_1}$**, 0).**

If 0 were not in the interior of **R(** $\boldsymbol{t_1}$**, 0),** then it is on the boundary.

Hence from preceding argument it implies that

$$0 = c^T \int_0^{t_1} \left[\int_{-h}^0 X(t_1, s-\theta) d_\theta H^*(s-\theta,\theta)\right] ds.$$

So that

$$c^T \left[\int_{-h}^0 X(t_1, s-\theta) d_\theta H^*(s-\theta,\theta)\right] = 0, \text{ almost everywhere (a.e)}$$

This, by definition of properness implies that the system is not proper, since $c^T \neq 0$.

However, if 0 **ϵ** interior **R (t, 0)** for $t_1 > 0$

$$c^T \left[\int_{-h}^0 X(t_1, s-\theta) d_\theta H^*(s-\theta,\theta)\right] = 0, \text{ implies that } c = 0,$$

which is the properness of the system and by the equivalence in theorem 4.1, the relative controllability of system (4.1.1) on [0, t_1] ; $t_1 > 0$ is proved.

4.2. OPTIMALITY CONDITIONS FOR THE LINEAR NEUTRAL VOLTERRA INTEGRO-DIFFERENTIAL SYSTEMS.

The optimal control problem can best be understood in the context of a game of pursuit **(see Balachandran and Dauer (1996), Balachandran and Ananhdi (2003)).** The emphasis here is the search for a control energy that can steer the state of the system of interest to the target set (which can be a moving point function or a compact set function) in minimum time. In other words, the optimal control problem is stated as follows:

If $\mathbf{t}^*$ = infimum $\{ t : \mathbf{A(t,0)} \cap \mathbf{G(t,0)} \neq \phi$ for $t \in [0, t_1], t_1 > 0 \}$,

then there exists an admissible control $\mathbf{u}^*$ such that the solution of the system with this admissible control is steered to the target. The proposition that follows illustrates this assertion.

Proposition 4.1. 1

Consider the system (4.1.1) as a differential game of pursuit

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t,s)x(s)ds - g(t)\right] = A(t)x(t) + \int_0^t G(t,s)x(s)ds$$
$$+ \int_{-h}^0 d_\theta H(t,\theta)u(t+\theta),$$

with its basic assumptions. Suppose **A(t, 0)** and **G(t, 0)** are compact set functions then there exists an admissible control such that the state of the weapon for the pursuit of the target satisfies the system (4.1.1) if and only if

$\mathbf{A(t,0)} \cap \mathbf{G(t,0)} \neq \phi$.

Proof

Let $\{ u^n \}$ be a sequence in U. Since the constraint control set U is compact, then the sequence $\{ u^n\}$ has a limit u, as n tends to infinity. That is, $\lim_{n\to\infty} u^n = u$

Suppose the state z (t) of the weapon for pursuit of the target satisfies the system (4.1.1) on the time interval $[0, t_1]$, then $z(t) \in \mathbf{G(t,0)}$, for $t \in [0, t_1]$. We are to show that there exists $x(t,u) \in \mathbf{A(t,0)}$,for $t \in [0, t_1]$ such that $z(t) = x(t,u)$ for some $u \in U$.

Now $x(t, x_0, u^n) \in \mathbf{A(t,0)}$ and from

$$x(t, x_0, u^n) = X(t,0)[x(0) - g(0)] + g(t) - \int_0^t [\tfrac{\partial}{\partial t}] X(t,s)g(s)ds$$
$$+ \int_{-h}^0 dH_\theta \int_\theta^0 X(t, s-\theta)H(s-\theta,\theta)u_0{}^n(s)ds$$

$$+ \int_0^t \left[\int_{-h}^0 X(t, s-\theta)\, d_\theta H^*(s-\theta,\theta)\right] u^n(s)\, ds . \qquad (4.1.4)$$

Taking limit on both sides of (4.1.4), we have

$$\lim_{n\to\infty} x(t, x_0, u^n) = X(t,0)\big[x(0) - g(0)\big] + g(t) - \int_0^t [\tfrac{\partial}{\partial t}] X(t,s) g(s) ds$$

$$+ \left[\int_{-h}^0 dH_\theta \int_\theta^0 X(t, s-\theta) H(s-\theta,\theta)\right] \lim_{n\to\infty} u_0{}^n(s)\, ds$$

$$+ \int_0^t \left[\int_{-h}^0 X(t, s-\theta)\, d_\theta H^*(s-\theta,\theta)\right] \lim_{n\to\infty} u^n(s)\, ds , \text{ for } t \in [0, t_1].$$

$$x(t, x_0, u) = X(t,0)[x(0) - g(0)] + g(t) - \int_0^t [\tfrac{\partial}{\partial t}] X(t,s) g(s) ds$$

$$+ \int_{-h}^0 dH_\theta \int_\theta^0 X(t, s-\theta) H(s-\theta,\theta) u_0(s) ds$$

$$+ \int_0^t \left[\int_{-h}^0 X(t, s-\theta)\, d_\theta H^*(s-\theta,\theta)\right] u(s)\, ds = x(t, x_0, u) \in \mathbf{A}(t, 0),$$

since **A(t ,0)** is compact and $\lim_{n\to\infty} x(t, x_0, u^n) = x(t, x_0, u)$.

Thus, there exists a control u ϵ U such that $x(t_1, x_0, u) = z(t_1)$, for $t_1 > 0$.

Since $z(t_1)$ ϵ **G(** t_1 **,0)** and also, is in **A(**t_1**,0)**, it follows that

$$\mathbf{A}(t, 0) \cap \mathbf{G}(t, 0) \neq \phi, \qquad \text{for } t \in [0, t_1].$$

Conversely,

Suppose that the intersection condition holds, i.e. **A(t ,0)** ∩ **G**(t,0) ≠ ф,

t ϵ [0, t_1], then there exists z(t) ϵ **A**(t ,0) such that z(t) ϵ **G(t ,0)** .

This implies that z(t) = x (t, x_0,u) and hence establishes that the state of the weapon of pursuit of the target satisfies the system (4.1.1). This completes the proof.

Remark 4.1:

The above stated and proved proposition in other words states that in any game of pursuit described by a linear neutral volterra integro-differential equation, it is always possible to obtain the control energy function to steer the system state to the target in finite time. The next theorem is, therefore, a consequence of this understanding and provides sufficient conditions for the existence of the control that is capable of steering the state of the system (4.1.1) to the target set in minimum time.

4.3. EXISTENCE OF AN OPTIMAL CONTROL.

The theorem below shows that the controllability of a system guarantees the existence of its optimal control.

Theorem (4.2)

Consider the system (4.1.1), that is,

$$\frac{d}{dt}\left[x(t)-\int_0^t C(t,s)x(s)ds-g(t)\right]=A(t)x(t)+\int_0^t G(t,s)x(s)ds$$
$$+\int_{-h}^{0} d_\theta H(t,\theta)u(t+\theta) \qquad (4.1.1)$$

with its basic assumptions.

Suppose the system (4.1.1) is relatively controllable on the finite interval [0, t_1] ,then there exists an optimal control.

Proof

By the controllability of the system (4.1.1), the intersection condition holds. That is, **A(t ,0)** ∩ **G(t ,0)** ≠ ϕ. Also, x (t, x_0,u) ϵ **G**(t_1,0) , so $z(t_1)$ = x (t_1,x_0, u). Recall that the attainable set **A(t , 0)** is a translation of the reachable set **R(t , 0)** through η which is given as

$$\eta = X(t,0)[x(0)-g(0)]+g(t)-\int_0^t\left[\frac{\partial}{\partial t}\right]X(t,s)g(s)ds$$
$$+\int_{-h}^{0} dH_\theta \int_\theta^0 X(t,s-\theta)H(s-\theta,\theta)u_0(s)ds\,.$$

It follows that z (t) ϵ **R (**t_1**, 0)** for t ϵ [0, t_1] , t_1 > 0 and can be defined as

$$z(t)=\int_0^t\left[dH_\theta\int_{-H}^{0}X(t,s-\theta)d\theta H^*(s-\theta,\theta)\right]u(s)ds\,.$$

Let t^* = infimum $\{$ t : z(t) ϵ **R(t , 0)** , t ϵ [0 ,t_1] $\}$.

Now $t_1 \geq 0$ and there is a sequence of times { t_n } and corresponding sequence of controls {u^n} ⊂U with { t_n} converging to t^* (the minimum time)

let $z(t_n)$ = y(t_n, u^n) ϵ **R(**t_1**,0).** Also

$$|z(t^*)-y(t^*,u^n)| = |z(t^*)-z(t_n)+z(t_n)-y(t^*u^n,)|$$
$$\leq |z(t^*)-z(t_n)|+|z(t_n)-y(t^*,u^n)|$$
$$\leq |z(t^*)-z(t_n)|+|y(t_n,u^n)-y(t^*,u^n)|$$
$$\leq |z(t^*)-z(t_n)|+\int_{t^*}^{t_n} y(s)\,ds.$$

By the continuity of z(t) which follows the continuity of **R(t ,0)** as a continuous set function and the integrability of ‖y (t) ‖ ,it follows that

$$y(t^*u^n,)\rightarrow z(t^*) \quad \text{as} \quad n\rightarrow\infty \text{ , where } z(t^*)=y(t^*,u^*)\in \mathbf{R(t,0)}.$$

For some u^* ϵ U and by definition of t^*; u^* is an optimal control.

This establishes the existence of an optimal control for the linear neutral volterra integro-differential equation (4.1.1).

4.4. THE FORM OF THE OPTIMAL CONTROL

In this section, we shall derive the form of the optimal control of our system of interest and express same using the definition of the signum function.

Definition (signum)

The signum function is defined by sgnx = $\begin{cases} \mathbf{1, x > 0} \\ \mathbf{-1, x < 0} \end{cases}$

Theorem 4.3

Consider the linear neutral volterra integro-differential equation given as the system

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t,s)x(s)ds - g(t)\right] = A(t)x(t) + \int_0^t G(t,s)x(s)ds$$
$$+ \int_{-h}^0 d_\theta H(t,\theta)u(t+\theta) \qquad (4.1.1)$$

with its basic assumptions, u^* is the optimal control energy for system (4.1.1) if and only if u^* is of the form

$$u^*(t) = \text{sgn}\left[c^T \int_{-h}^0 X(t, s-\theta)d_\theta H^*(s-\theta,\theta)\right], \quad \text{where} \quad c \in E^n.$$

Proof

Suppose, u^* is the optimal control energy for the system (4.1.1), then it maximizes the rate of change of

$y(t, u) = \int_{-h}^0 X(t, s-\theta)d_\theta H^*(s-\theta,\theta)u(t)$, for $u \in U$; in the direction of c .Since u(t) are admissible controls, that is, they are constrained to lie in a unit sphere, we have

$$c^T \int_{-h}^0 X(t, s-\theta)d_\theta H^*(s-\theta,\theta)u(t) \le \left| c^T \int_{-h}^0 X(t, s-\theta)d_\theta H^*(s-\theta,\theta) \right|$$
$$\le c^T \int_{-h}^0 X(t, s-\theta)d_\theta H^*(s-\theta,\theta)\ \text{sign}\left[c^T \int_{-h}^0 X(t, s-\theta)d_\theta H^*(s-\theta,\theta)\right]$$

(4.1.5)

This inequality follows from the fact that, for any non-zero number v, then v ≤ sign v.

Hence defining

$$u^* = \text{sgn}\left[c^T \int_{-h}^0 X(t, s-\theta)d_\theta H^*(s-\theta,\theta)\right] \qquad (4.1.6),$$

The inequality (4.1.5), becomes

$$c^T \int_{-h}^0 X(t, s-\theta)d_\theta H^*(s-\theta,\theta)u(t) \le c^T \int_{-h}^0 X(t, s-\theta)d_\theta H^*(s-\theta,\theta)\ u^*(t)$$

This shows that the control that maximizes $y(t, u) \in R(t, 0)$ is of the form

$$u^* = \text{sgn}\left[c^T \int_{-h}^0 X(t, s-\theta)d_\theta H^*(s-\theta,\theta)\right].$$

Conversely,

Let $u^* = \mathrm{sgn}[c^T \int_{-h}^{0} X(t, s-\theta) d_\theta H^*(s-\theta, \theta)]$, then for the controls u ϵU

$$c^T \int_0^t [\int_{-h}^{0} X(t, s-\theta) d_\theta H^*(s-\theta, \theta)] u(s) ds$$

$$\leq c^T \int_0^t [\int_{-h}^{0} X(t, s-\theta) d_\theta H^*(s-\theta, \theta)] \; \mathrm{sgn}[c^T \int_{-h}^{0} X(t, s-\theta) d_\theta H^*(s-\theta, \theta)] ds$$

$$\leq c^T \int_0^t [\int_{-h}^{0} X(t, s-\theta) d_\theta H^*(s-\theta, \theta). 1] ds$$, since for v ≠o, sgn v > 0

$$\leq c^T \int_0^t [\int_{-h}^{0} X(t, s-\theta) d_\theta H^*(s-\theta, \theta) u^*(s)] ds \quad .$$

This shows that u^* maximizes $\int_0^t [\int_{-h}^{0} X(t, s-\theta) d_\theta H^*(s-\theta, \theta)] u(s) ds$

over all admissible controls u ϵ U.

Hence u^* is an optimal control for system (4.1.1). This completes the proof.

Remark 4.2

It is evident from theorem 4.3, that if u^* is the optimal control then the target is on the boundary of the reachable set. To see this, let

$y^* = y(t^*, u^*) = \int_0^t [\int_{-h}^{0} X(t, s-\theta) d_\theta H^*(s-\theta, \theta)] u^*(s) ds$; for t^* ϵ $[0, t_1]$

$y = y(t, u) = \int_0^t [\int_{-h}^{0} X(t, s-\theta) d_\theta H^*(s-\theta, \theta)] u(s) ds$; for t ϵ $[0, t_1]$

Then, from the result of theorem 4.3

$c^T y \leq c^T y^*$ *implies* $c^T (y - y^*) \leq 0$, for y ϵ R(t, 0).

Since the reachable set **R(t , 0)** is closed, convex subset of E^n, there is a support plane π of **R(t , 0)** through c with $c \neq 0$ an outward normal to π at y^* and hence y^* is in the boundary of the reachable set. This realization is formalized with the next theorem.

Theorem 4.4

Let u^* be the optimal control for the system (4.1.1) with t^* the minimum time, then the target $z(t^*) = x(t^*, x_0, u^*)$ is on the boundary of the attainable set **A(t, 0).**

i.e $z(t^*)$ ϵ ∂**A(t, 0)** . (where ∂ symbolizes boundary); for t, t^*ϵ $[0, t_1]$.

Proof

Suppose $u^*$is an optimal control, then $x(t^*, u^*) = (\eta + y^*)$ ϵ $R(t^*, 0)$.

Therefore $x(t^*, u^*)$ ϵ **A(t^*, 0).**

Now suppose for contradiction $x(t^*, u^*)$ is not on the boundary, then $x(t^*, u^*)$ is in the interior of **A(t^*, 0);** t^*ϵ $[0, t_1]$.

Therefore there is a ball $B(x(t^*, u^*), r)$ centered at $x(t^*, u^*)$,radius r in **A(t^*,0).**

Because **A(t, 0)** is a continuous set function of t, we can preserve the above inclusion for t near t^* . If we reduce the size of the ball B(x (t^*, u^*) , r); that is, if there is an ε > 0 such that $B(x(t^*, u^*), \frac{r}{2}) \subset$ **A(t , 0)** for $t^* - \epsilon \leq t \leq t^*$.

Thus, $x(t^*, u^*) \in$ **A(t , 0)** for $t^* - \epsilon \leq t \leq t^*$.

This of course contradicts the optimality of t^* and u^*as the optimal control.

This contradiction however, proves that the target z (t^*) is on the boundary of the attainable set **A(t^*,0)** and hence on the boundary of the reachable set **R(t^*,0)**

Remark 4.3:

This theorem is the basis of Pontrygin's Maximum Principle **(see Balachandran and Dauer (1998)).**There are other fascinating properties that emanate from the convexity property of the reachable set in **Balachandran, Balasubramaniam and Dauer (1995) .**It is stated that if the reachable set of a system is strictly convex, then the system is said to be normal and optimal control for such a system is said to be Bang-Bang. By the Bang-Bang principle, any point of the reachable set that can be reached by an admissible control can be reached by a Bang-Bang control. Following the arguments in **Balachandran, Balasubramaniam and Dauer (1995), Chukwu (2001), Balachandran and Anandhi (2003)**, the above results, can easily be proved for the system (4.1.1).

4.5. UNIQUENESS OF OPTIMAL CONTROL

Here, a new method of approach is derived for the proof of the existence of optimal control.

Theorem 4.5

Consider the **system (4.1.1)** with its standing hypothesis. Suppose $u^*$is the optimal control, then it is unique.

Proof

Let u^* and v^* be optimal controls for the system (4.1.1), then u^* and v^*maximize

$$c^T \int_0^t [\int_{-h}^0 X(t, s-\theta) d_\theta H^*(s-\theta, \theta)\,] , \quad \text{for } t \in [0, t_1],\ t_1 > 0;$$

over all admissible controls u ϵ U, and so we have the inequality with u^*as the optimal control.

$$c^T \int_0^t [\int_{-h}^0 X(t, s-\theta) d_\theta H^*(s-\theta, \theta)\,] u(s) ds$$
$$\leq c^T \int_0^{t^*} [\int_{-h}^0 X(t, s-\theta) d_\theta H^*(s-\theta, \theta)\,] u^*(s) ds \qquad (4.1.7)$$

Also, using v^* as optimal control, we have

$c^T \int_0^t [\int_{-h}^0 X(t, s-\theta) d_\theta H^*(s-\theta, \theta)\ ] u(s) ds$

$$\leq\ c^T \int_0^{t^*} [\int_{-h}^0 X(t, s-\theta) d_\theta H^*(s-\theta, \theta)\ ] v^*(s) ds$$

(4.1.8)

Taking maximum of u, over [-1, 1], the range of definition of u^*in (4.1.7) and (4.1.8), we have the equations

$$c^T \int_0^t \left[\int_{-h}^0 X(t, s-\theta) d_\theta H^*(s-\theta, \theta)\right] \max. |u(s)| ds\ , \quad \text{for } -1\leq s \leq 1$$

$$= c^T \int_0^{t^*} [\int_{-h}^0 X(t, s-\theta) d_\theta H^*(s-\theta, \theta)\ ] u^*(s) ds\ , \quad \text{for } u, u^* \epsilon\ U \qquad (4.1.9)$$

Also ,

$$c^T \int_0^t [\int_{-h}^0 X(t, s-\theta) d_\theta H^*(s-\theta, \theta)\ ] \max \big| u(s) \big| ds$$

$$=\ c^T \int_0^{t^*} [\int_{-h}^0 X(t, s-\theta) d_\theta H^*(s-\theta, \theta)\ ] v^*(s) ds \qquad (4.2.0)$$

for u, $v^*\epsilon$ U, v^* being optimal and -1≤ s ≤1.

Subtracting equation (4.19) from equation (4.2.0), we have

$$0 = c^T \int_0^{t^*} [\int_{-h}^0 X(t, s-\theta) d_\theta H^*(s-\theta, \theta)\ ]\{\ u^*(s) - v^*(s)\ \} ds\ , \qquad \text{Implies that}$$

$$0 = [\int_{-h}^0 X(t, s-\theta) d_\theta H^*(s-\theta, \theta)\ ]\{\ u^*(s) - v^*(s)\ \}$$

$u^*(s) - v^*(s) = 0$, implies that $\quad u^*(s)\ =\ v^*(s)$

This establishes the uniqueness of the optimal control for the system (4.1.1).

4.6. GLOBAL UNIFORM ASYMPTOTIC STABILITY FOR NONLINEAR INFINITE NEUTRAL DIFFERENTIAL SYSTEMS

Preamble.

Consider the system (1.1.8),

$$\frac{d}{dt}[D(t, x_t)] = L(t, x_t)x_t + f(t, x_t) + \int_{-\infty}^0 A(t, \theta) x(t+\theta) d_\theta : x_{t_0} = \phi.$$

(a nonlinear infinite neutral system)

Consider the system below,

$$\frac{d}{dt}[D(t, x_t)] = L(t, x_t)x_t + f(t, x_t) + \int_0^\infty A(t, \theta) x(t+\theta) d_\theta \qquad (4.2.1)$$

(circularity of the function from -∞ to 0, and from 0 to ∞).

We can linearize system (4.2.1) as in **Chukwu (1992)** by setting x_t = z in L; a specified function inside the function $L(t, x_t)x_t$ to have L(t, z) x_t with no loss of generality.

Thus system (4.2.1) becomes

$$\frac{d}{dt}[D(t,x_t)] = L(t,z)x_t + \int_0^{\infty} A(t,\theta)\, x(t+\theta)d_\theta + f(t,x_t) \quad (4.2.2)$$

Evidently

$$L(t,z)x_t = \sum_{k=0}^{\infty} A_k x(t-w_k) + \int_{-\infty}^{0} A(t,\theta)\, x(t+\theta)d_\theta + \int_0^{\infty} A(t)\, x(t+\theta)d_\theta$$

$$L^*(t,z)x_t = \sum_{k=0}^{\infty} A_k x(t-w_k) + \int_{-\infty}^{\infty} A(t,\theta)\, x(t,\theta)d_\theta$$

The representations L, L* are the same under the following assumptions

$$L(t,z)x_t = \lim_{p\to\infty} \sum_{k=0}^{p} A_k x(t-w_k) + \lim_{M,N\to\infty} \int_M^N A(t,\theta)\, x(t+\theta)ds$$

We assume the limits exist, giving finite partial sum for the infinite series and the improper integrals. Thus the system

$$L^*(t,z)x_t = \sum_{k=0}^{\infty} A_k x(t-w_k) + \int_{-\infty}^{\infty} A(t,\theta)\, x(t+\theta)d_\theta$$

is finite and well defined function. In the light of the above, the system (4.2.1) reduces to

$$\frac{d}{dt}[D(t,z)x_t] = L(t,z)x_t + f(t,x_t);\ x(t_0) = \phi \epsilon C \quad (4.2.3)$$

where $L(t,z)x_t = \sum_{k=0}^{p} A_k\, x(t-w_k) + \int_{-h}^{0} A(t,\theta)\, x(t+\theta)d\theta$

Integrating (4.2.3), after linearizing, we have the solution

$$x(t) = x(t,t_0,\phi,0) + \int_0^t X(t,s)\, f(s,x_s)ds \quad (4.2.4)$$

where X(t ,s) is the fundamental matrix of the homogenous part of the system (4.2.3).

X (t , s) = I (identity matrix); t = s

From the transformation in **Hale (1977)** there is a linear operator T such that

$$X_t(s)\phi = T(t,s)X(\theta);\ \theta \epsilon [-h, 0]. \quad (4.2.5)$$

$$X(t+\theta, s) = T(t,s)X(\theta),$$

For $\theta = 0$, X(t, s) = T(t, s).I = T(t, s), where T is defined as follows:

(i) T(t, s) is an operator defined on **C** = **C** ([-h, 0], E^n)

(ii) T(t, s) is bounded for T ϵ C.

(iii) T(0) = I and T is strongly continuous.

(iv) T(t, s) is completely continuous in t.

(v) The family **{T(t, s) for t > s }** is a semi group of linear transformations **(see Asuquo and Usah (2008)** for these properties. Now writing (4.2.4) in terms of T (t, s), we have

$$x(t,t_0,\phi,f) = [\,T(t,t_0)\,]\phi(0) + \int_{t_0}^{t} X(s,x_s)\, ds \quad (4.2.6)$$

4.6.1. RESULTS ON STABILITY ON NEUTRAL SYSTEMS.

A homogenous linear neutral equation is given by

$$\frac{d}{dt}[D(t, x_t)] = L(t, x_t) \quad (4.2.7)$$

$D(t, x_t)$ is called the functional difference operator. We now give the condition for the uniform stability of the functional difference operator.

PROPOSITION 4.7

Suppose , A_k, k =1, 2,…,N are constant matrices and τ_k, where $0 < \tau \leq h$ are real valued number such that $\frac{i}{k}$ are rational if N >1. If

$$D\phi = \phi(0) - \sum_{K=1}^{N} A_k \phi(- \quad _k)$$

and all roots of the equation $\det [I - \sum_{k=1}^{N} A_k e^{- \quad k}] = 0$, have modulus less than one, then D is uniformly stable.

Proof: **(see Artstein and Tradmar (1982).**

THEOREM 4.6. (this is the condition for uniform stability of the system (4.2.7)).

Consider the system (4.2.7)

$$\frac{d}{dt}[D(t, x_t)] = L(t, x_t) \quad ,x_{t_0} = \phi \ . \quad (4.2.8)$$

Suppose u(t), v(t), w(t) are continuous functions and u(t), v(t) are positive non-decreasing for $t > 0$ and $u(0) = v(0) = 0$;

w(t) is non-negative and non-decreasing and $v : (\tau, \infty) \times \mathbf{C} \rightarrow E^n$ is continuous function satisfying: (i) $u(|D(t)\phi|) \leq v(t, \phi) \leq v(\|\phi\|)$

(ii) $\dot{v}(t, \phi), \leq -w(|D(t)\phi|$

(iii) If D is uniformly stable with respect to $\phi \in \mathbf{C}([y, \infty), E^n)$,then the trivial solution of (4.2.7) is uniformly stable. If in addition, $w(t) > 0$ for $t > 0$, then the trivial solution is uniformly asymptotical stable.

Proof: **(see Artstein (1982)).**

4.6.2. RESULTS ON THE EXPONENTIAL ASYMPTOTIC STABILITY IN THE LARGE FOR NONLINEAR NEUTRAL SYSTEMS.

Theorem 4.8. (Exponential Asymptotic Stability in the Large).

Consider the system (4.2.2)

$$\frac{d}{dt}[D(t, x_t)] = L(t, z)x_t + \int_0^{\infty} A(t, \theta)\, x(t + \theta) d_\theta \quad (4.2.8)$$

and its perturbation

$$\frac{d}{dt}[D(t,x_t)] = L(t,z)x_t + \int_0^{\infty} A(t,\theta)\, x(t+\theta)d_\theta + f(t,x_t) \qquad (4.2.9)$$

under the hypothesis on the system(4.2.9) with the following assumptions:

(i) D(t, ϕ) = D(t) ϕ and is uniformly stable.

(ii) f is continuous and $f = f_1 + f_2$ satisfies the conditions

f_1 (t , ϕ) ≤ v(t)│D(t , ϕ) │; v(t) ≥ 0 , ϕ ϵ **C**, $\int_0^{\infty}$ v(t)dt < ∞.

Also, for ε > 0,

│f_2(t , ϕ) │ ≤ ε│D(t, ϕ)│ , t ≥ 0, ϕ ϵ C

then ,the solution $x_t(t_0$, ϕ) of the system(4.2.9) is exponentially asymptotically stable, in the large.

Proof

By condition (i) and the assumption on a continuous Lyapunov function v(t ,ϕ) defined in $[t_0, \infty)$x**C** satisfying the following conditions:

$$│ D(t, ϕ) │ ≤ v(t, ϕ) ≤ M‖ϕ ‖ \qquad (4.3.0)$$

$$\dot{v}(t, ϕ) \quad ≤ \quad av(t, ϕ) \qquad (4.3.1)$$

for all t ≥ t_0. a,M positive constants as in **Balachandran and Dauer (1986)**, the homogeneous part of system (4.2.3) vis-a-vis system (4.2.8) is uniformly asymptotically stable. That is, the solution $x_t(t,$ ϕ) of system (4.2.8), satisfies

$$‖x_t(t_0, ϕ) ‖≤ k\, e^{a(t-t_0)}‖ ϕ ‖ . \qquad (4.3.2)$$

We now show that the solution $x_t(t_0$, ϕ) of system (4.2.3) vis-a-vis (**4.2.9**) is uniformly exponentially asymptotically stable in the large.

Here $\dot{v}$ is the usual upper right hand derivative along the solution path of system (**4.2.9**).

Let $x_{t_0}(t_0$, ϕ) and $x_t(t_0$, ϕ) be the solutions of systems (4.2.8) and (4.2.9) respectively with initial value ϕ at t_0.

From relations (4.3.0) and (4.3.1), we have

$$\dot{v}_{4.2.9}(t, ϕ) ≤ \dot{v}_{4.2.8}(t_0, ϕ) + M \lim_{h\to 0} \frac{1}{h}\left[\ddot{x}_{t_{0+h}}(t_0, ϕ) - \dot{x}_{t_{0+h}}(t_0, ϕ)\right] \qquad (4.3.3)$$

on the alternative,

$$D_{t_{0+h}}(\ddot{x}_{t_{0+h}} - \dot{x}_{t_{0+h}}) = \int_{t_0}^{t_{0+h}}[L(t,z)\ddot{x}_t - L(t,z)\dot{x}_t + f(t,\ddot{x}_t]ds \qquad (4.3.4)$$

by the non-atomicity of D, the functional difference operator at zero means there exists a constant h_0 such that

$\| \ddot{x}_{t_{0+h}} - \dot{x}_{t_{0+h}}) \| \leq \frac{1}{1-r(h_0)} \int_{t_0}^{t_0+h} [L(t,z)\ddot{x}_t - L(t,z)\dot{x}_t + f(t,\ddot{x}_t]ds$ (4.3.5)

for a function r. We use the inequality (4.3.3) to obtain

$\dot{v}_{4.2.9}(t, \phi) \leq \dot{v}_{4.2.8}((t_0, \phi) + N[f_1(t, \phi) + f_2(t, \phi)]$ (4.3. 6)

For all t ≥ 0 , ϕ ϵ C, where

$N = \max [\frac{m}{1-r(h_0)}, M]$

We choose **ε** > 0 so that $\varepsilon = \frac{a}{2N}$

In this domain, define a function

$$W_{4.2.9}((t, \phi) = v(t, \phi)\exp[N \int_0^t v(s)\, ds \quad (4.3.7)$$

The derivative along the solution path is

$\dot{w}(t_1,x_t) = \dot{v}_{4.2.9}((t_1,x_t) \exp\{- N \int_0^t v(s)\, ds \} + v(t_1,x_t) [-NV(t)\exp \{-N \int_0^t v(s)\, ds \}]$.

From (4.3.1) and (4.3.6), we have

$\dot{w}(t_1,x_t) \leq \{\exp (-N \int_0^t v(s)\, ds \} \{ - N v(t) v(t,x_t) - av(t,x_t)$

$+ N(| f_1(t,x_t) | + | f_2(t,x_t) |)\}$.

From condition (ii) of the theorem 4.8, we have

$\dot{W}_{4.2.9}((t, \phi) \leq \{ \exp (-N \int_0^t v(s)\, ds)\} \{ -N v(t) v(t,x_t) - av(t,x_t)$

$+ \varepsilon N | D(t)x_t | + Nv(t) | D(t) x_t | \}$

Taking the value of $| D(t)x_t |$ as in (4.3.0), we have

$\dot{W}_{4.2.9}((t_1,x_t) = \exp\{ - N\int_0^t v(s)\, ds \} v(t,x_t) [Nv(t) – a + Nv(t) + N\varepsilon]$

From (4.3.7), we have

$$\dot{W}_{4.2.9}(t_1,x_t) = W(t,x_t) (N\varepsilon- a) = W(t,x_t)(- \frac{a}{2}),$$

This is clearly from our choice of ε.

implies $\frac{d}{dt}(t_1,x_t) = W(t,x_t)(- \frac{a}{2}), \Rightarrow \frac{d}{W(t,x_t)} W(t_1,x_t) = (- \frac{a}{2})dt,$

Integrating, we have

$\text{Log}_e W_{4.2.9}(t_1,x_t) = - \frac{at}{2}, \quad t \geq t_0.$ This implies that

$W(t_1,x_t (t_0, \phi)) \leq \exp\{ \frac{-a(t-t_0)}{2} \}$, from which we obtain

$D(t) x_t (t_0, \phi) \leq \beta e^{c(t-t_0)} \| \phi \|$,for $t \geq t_0$, ϕ ϵC, where

$\beta = M\exp(-N \int_0^\infty V(t)dt), c = -\frac{a}{2}$.

This establishes the uniform exponential asymptotic stability in the large of (4.2.2) vis-a-vis the system (4.2.9)

REMARK 4.8

Observe that the system (4.2.9) is uniformly exponential asymptotically stable in the large implies that the solution of the large implies the solution of the system has an exponential estimate. That is,

$\| x_t\ (t_0, \phi) \| \leq \beta\ e^{c(t-t_0)} \|\phi\|$. Clearly, as $t \to \infty$, $\| x_t\ (t_0, \phi) \| \to 0$

Since the trivial solution has been shown to be uniformly stable, the system (4.2.9) is said to be uniformly asymptotically stable. Evidently, uniform exponential asymptotic stability in the large of a system guarantees its uniform asymptotic stability. We now state the realization as a theorem.

Theorem 4.9. (Global Uniform Asymptotic Stability).

Consider the nonlinear neutral system

$$\frac{d}{dt}[\, D(t, x_t)\,] = L(t, z)\, x_t + f(t, x_t) + \int_{-\infty}^{0} A(t)\, x(t+\theta)d\theta \ ; \ x_{t_0} = \phi$$

Under all the assumptions of theorem 4.8, assume further that the system (4.2.9) satisfies

$$f(t, 0) = 0,$$

then the system (4.2.9) is uniformly exponentially asymptotically stable.

Proof

We seek to prove that system (4.2.2) vis-à-vis (4.2.9) is stable and for every $\eta > 0$, every $t_0 \geq 0$ and every number H_0 ,there exists a $T(\eta)$ independent of t_0 such that $\phi \in \mathbf{C}$, $\|\phi\| < H_0$, implies

$\|x_t\ (t_0, \phi)\| < \eta$, for $t > t_0 + T(\eta)$.

By the assumption of uniformly exponentially asymptotic stability in the large of the system (4.2.9), we have

$$|D(t)\, x_t\ (t_0, \phi)| \leq h(t)\beta e^{a(t-t_0)} \|\phi\| \qquad (4.3.8)$$

for $t \geq t_0$, $\phi \in C$.

From **Lemma 3.4** of **Cruz and Hale (1970),** there are positive constants a, b, c, d so that for any $s \in [t_0, \infty)$.

$$\| x_t\ (t_0, \phi) \| \leq e^{a(t-s)}[b\beta\|\phi\| + c\beta\|\phi\| + d\beta e^{a(t-t_0)}\|\phi\|] \qquad (4.3.9)$$

for $t > s + h \geq t_0 + h$.

Let $\boldsymbol{\varepsilon} > 0$, be a given constant.

Choose $\delta = (b\beta + c\beta + d\beta) < \boldsymbol{\varepsilon}$, then if

$\|\phi\| \leq \delta \|x_t(t, \phi)\| \leq \beta b\delta + c\beta\delta + d\beta\delta < \boldsymbol{\varepsilon}$

This proves the uniform stability if $x = 0$ is a solution.

It now remains to show that

$$\| x_t(t_0, \phi) \| \le \frac{\eta}{2}, \text{ for } t \ge T + t_0$$

where т does not depend on t_0 , proves the global uniform asymptotic stability of the system (4.2.3) vis-à-vis the system (4.2.9)

Let η, H_0 be arbitrary positive numbers, choose $T_1 = T_1(\eta)$ so that

$$d \sup | h(u) | < \frac{\eta}{2}, \; ; \; T(\eta) \le u \tag{4.4.0}$$

and

choose T = T(η, H_0) $\ge T_1$ + h so that

$$\exp\{ -a(T - T_1 +)\}\{ bH_0 + c \max_{T<U} |h(u)| \le \frac{\eta}{2}, \tag{4.4.1}$$

For any $t_0 \epsilon [T, \infty)$ and s = t_0 + T if $\|\phi\| \le H_0$,

the relation (4.3.9) implies

$$\|x_t(t_0, \phi)\| \le \exp\{-a(t - t_0 - T_1 +)[\mathbf{b\delta} + \mathbf{c} \max_{T<U} |h(u)| + \mathbf{d} \max_{t_0+T-\eta \le u} |h(u)|] \}$$

From (4.4.0) and (4.4.1)

$$\| x_t(t_0, \phi)\| \le \frac{\eta}{2}\{\exp(-a(t - T - t_0,) + \frac{\eta}{2}\} \; , \text{ for } t \ge t_0, + T_I Y + h .$$

Therefore

$$\| x_t(t_0, \phi) \| \le \frac{\eta}{2} \; ; \; t \ge T + t_0, \quad , \text{ as } t \to \infty$$

Since T does not depend on the initial time t_0 , system (4.2.3) vis-à-vis (4.2.9) is globally uniformly asymptotically stable.

REMARK 4.9

It is evident that under boundedness conditions on the perturbation function, the uniform asymptotic stability of the non-linear base guarantees the global uniform asymptotic stability of the non-linear perturbation. To shed light on theorem 4.8, we give the following example(application).

APPLICATION

Consider the system

$$D(t, x_t) = L(t, x_t) + f(t, x_t) \tag{4.4.2}$$

where D(t, x_t) = $x(t) - cx(t-h)$ and a ,c, are constants such that a > 0, |c| < 1.

L(t , x_t) $= ax(t)$.

$$f(t, x_t) = \{ \frac{1}{(1+t^2)} [\sin(1 + (x-1))] + \log_e[e^{\varepsilon}(\cos(3 - 2(x-1))]\}.$$

We wish to discuss the exponential stability of the system using the Lyapunov function

$$v(\theta) = (D\phi)^2 + ac\int_{-h}^{0} \Phi^2(\theta)d\theta$$

and infer from this result that the system is uniformly asymptotically stable.

Solution

We first show that the functional difference operator

D(t, x_t) $= x(t) - cx(t-h)$ is uniformly stable.

Now,

Dϕ = ϕ(0) – **c** ϕ(-h)

The number of delays is N= 1, from proposition **4.7,** $\frac{j}{k} = 1$, and hence

$$\det[\, I - \sum_{k=1}^{N} A_k \xi^{-h}\,] = 0 \;,\Rightarrow\; \det[\, I - c\xi^{-h}\,] = 0$$

$$\Rightarrow \quad c\xi^{-h} = 1$$

$$\Rightarrow \quad \xi^{-h} = \frac{1}{c} \;, \quad \Rightarrow \quad \xi^{h} = c\;,$$

$$\Rightarrow \quad |\xi| = |c|^{\frac{1}{h}} < 1\;, \quad \text{since } |c| < 1.$$

We have shown that the modulus of c is less than one, hence by the provision of the proposition, we conclude that the functional difference operator D(t, x_t) is uniformly stable.

A resort to **Cruz and Hale (1970)** establishes the uniform asymptotic stability of the homogenous system.

$$\frac{d}{dt}[D(t, x_t)] = L(t, x_t).$$

We are now in a good position to investigate the asymptotic exponential stability of the non linear system

$$\frac{d}{dt}[D(t, x_t)] = L(t, x_t) + f(t, x_t).$$

$$f(t, x_t) = \{ \frac{1}{(1+t^2)}[\sin(1 + (x-1))] + \log_e[\, e^{\varepsilon}(\cos(3 - 2(x-1))]\} : \varepsilon > o$$

$$f_1(t, x_t) = \frac{1}{(1+t^2)}[\sin(1 + (x-1))]$$

$$f_2(t, x_t) = \log_e[\, e^{\varepsilon}(\cos(3 - 2(x-1))]$$

Evidently,

$$D(t, \phi) = a(t)\,, \; |D(t, \phi)| = |a(t)|$$

$$|D(t, \phi)| \le k\|\phi\|\;, \; k > 0.$$

$$|f_1(t, \phi)| \le \frac{1}{(1+t^2)} \le v(t)|D(t, \phi)|\;, \; v(t) \ge 0$$

Set

$$V(t) = \frac{1}{(1+t^2)}\;, \; t_0 = 0$$

$$\int_0^\infty \frac{dt}{(1+t^2)} = [tan(t)]_0^\infty = \pi < \infty.$$

$$|f_2(t,\phi)| = |\log_e e^{\varepsilon}(\cos(3-2(x-1))]\}| = |\log_e e^{\varepsilon}|\,|(\cos(3-2(x-1))]\}|$$

$$\leq \log_e e^{\varepsilon} = \varepsilon \log_e e = \varepsilon.$$

Evidently,

$f_2(t,\phi) < \boldsymbol{\varepsilon}|D(t,\phi)|$: $t \geq 0$, $\phi \in C$ .Having satisfied the boundedness condition on the perturbation function, we use the given Lyapunov function to show that the base system is uniformly asymptotically stable.Given

$v(\theta) \geq (D\Phi)^2 + ac^2 \int_{-h}^{0} \phi^2(\theta)d\theta$. Clearly, $v(\theta) \geq (D\Phi)^2$.

This shows that a lower bound exists.

We now compute the derivative of v

$$\dot{v}(\theta) = 2\,(D\,\phi)\,\frac{d}{dt}(D\,\phi) + ac^2\,[\phi^2(0) + \phi^2(-h)]$$

$$\leq 2[\phi\,(0) + c\,\phi\,(-h)\,][a\,\phi(0)] + ac^2\,\phi^2(0) + ac^2\,\phi^2(-h)$$

$$= 2a\,\phi^2(0) + ac\,\phi\,(-h)\,\phi\,(0) + ac^2\,\phi^2(0) + ac^2\,\phi^2(-h)$$

$$= a[\phi^2(o) + 2c\,\phi(-h)\phi(0) + c^2\phi^2(-h)\,] + a\,\phi^2(0) + ac^2\,\phi^2(0)$$

$$\leq a[\,\phi\,(0) + c\,\phi(-h)\,]^2 + a\,(1-c^2)\,\phi^2(0)$$

$$\dot{v}(\theta) = a[\,\phi\,(0) + c\,\phi(-h)\,]^2 + a(1-c^2)\,\phi^2(0)$$

$$\leq a(1-c^2)\,\phi^2(0) \leq ac\,\phi^2(0).$$

By the hypothesis on a, and c, $\dot{v}(\theta) \leq W(|D\,\phi|)$.

$\dot{v}(\theta)$ is therefore negative semi-definite. The homogenous system is uniformly asymptotically stable. By theorem 4.8, system (4.4.2) is exponentially asymptotically stable in the large. That is, the solution of (4.4.2) is such that for constants k, a

$$\|x_t(t_0,\phi)\| \leq \beta e^{a(t-t_0)}\|\phi\|, \quad \Rightarrow \quad \|D(x_t(t_0,\phi)\| \leq Ke^{a(t-t_0)}\|\phi\|$$

Observe that if $t \to \infty$, $\|x_t(t_0,\phi)\| \to 0$

This condition, in addition with the uniform stability of the functional difference operator, the linear base, establishes the uniform asymptotic stability of the non linear system.

4.7. EXISTENCE OF MILD SOLUTION OF NON-LINEAR NEUTRAL FUNCTIONALDIFFERENTIAL EQUATIONS IN BANACH SPACES

Preamble.

Recall that to extend the existence results for non-linear abstract neutral differential equations in the work of **Balachandran, Leelamani and Kim (2006)** to the system with delay base is the objective of our work here. We shall in this work, therefore present the system

$$\frac{d}{dt}\left[x(t) + g(t, x(t), x(u_1(t)), \ldots, x(u_m(t))\right] = L(t, x_t) + h(t, x(t), x(v_1(t)), \ldots, x(v_n(t)))$$

$x(\theta) = 0$; $\theta \in [-h, 0]$, $t \in J = [0, t_1]$, $t_1 > t_0$ (4.4.3),

with the purpose of obtaining **mild solutions** of system (4.4.3) in the Banach Spaces, using **Schaefer's fixed point theorem.(See Appendix A).**In the system (4.4.3), g, L, h are the systems parameters. L is the infinitesimal generator of compact analytic semi-group of bounded linear operators $T(t)$ in a Banach space X. E is the real line.

$$L(t, x_t) = \int_{-h}^{0} d\eta(., s, \phi)\, x(t+s) = \sum_{k=0}^{\infty} A_k(t - w_k) + \int_{-\infty}^{0} A(t, \theta) x(t + \theta)\, d_\theta$$

is a bounded linear operator, where the n x n matrix functions A_k, $A(t, \theta)$ are measurable in ((t, s) $\in$ E x E, $\theta \in [-\infty, 0)$. η is normalized such that

$\eta(t, s, \phi) = 0$, $s \geq 0$ for all ϕ

$\eta(t, s, \phi) = \eta(t, -h, \theta)$ for all $s \leq -h$

$\eta(t, s, \phi)$ is continuous from the left in s on $(-\infty, 0]$ and has bounded variations on $(-\infty, 0]$ for each t, ϕ and there is an integrable function M such that

$\| L(t, x_t) \| \leq M(t) \| x_t \|$,for all $t \in (-\infty, \infty)$, $\phi = x_t \in (-\infty, 0]$.

We assume $L(t, \phi)$ is continuous. Let $0 \in D(L)$, then the fractional power $L^{\mathbf{a}}$ for , $0 < a < 1$ as closed linear operator on its domain $D(L^{\mathbf{a}})$ is dense in X. Furthermore, $D(L^{\mathbf{a}})$ is a Banach space under the norm

$\| x \|_{\mathbf{a}} = \| L^{\mathbf{a}} x \|$ for all $x \in D(L^{\mathbf{a}})$ and it is denoted by X_a. h is a function defined on the product space $J x X^{n+1}$ into X

$g : [0, t_1] \times X^{n+1} \rightarrow X$, is a continuous function.

The delays $u_i(t)$, $v_j(t)$, are continuous scalar valued functions defined on J such that $u_i(t) \leq t$ and $v_j(t) \leq t$.

That is, these are values preceding t. We define the supremum norm on X by

$\| x \| = \max_{x \in X} |x(t)|$.

The imbedding $X_a \to X_b$ for 0< b < a < 1 is compact whenever the resolvent operator L is compact. For semi-group {T(t) }, the following properties will be used:

(i) There is a number $N_1 > 1$ such that $\|T(t)\| \le N_1$ for all t ϵ [0, t_1].

(ii) For any a > 0, there exists a positive constant N_2 such that

$$\|L^a T(t)\| \le \frac{N_2}{t^a}, \qquad 0 < t < \tau$$

To study system (4.4.3), we assume the hereditary property of the function.

Let x: (-∞, τ] → X,

x_t is a function defined on the delay interval (-∞ ,0] such that

$$x_t(\theta) = x(t+\theta)$$

belongs to some abstract phase space (-∞ ,0] .

In this work, the state space will be the abstract phase space C ([-∞, 0])

Definition 1.12.1: (see Balachandran, Leelamani and Kim (2006)).

A function $x(.)$ is called **a mild solution** of system (4.4.3) if

$$x(t) = 0 \quad , \quad \text{for } t \in (-\infty, 0].$$

The restriction of $x(t)$ to the interval [0, τ] is continuous and for each [0, τ], the function $x(t)$ satisfies system(4.4.3). That is, the function $x(t)$ satisfies the following integral equation :

$$x(t) = T(t)\{0 + g[0, x(u_1(0)),\ldots,x(u_m(0))] - g\{t, x(t), x(u_1(t)),\ldots,x(u_m(t))\}$$
$$-\int_0^t LT(t-s)\, g(s, x(s), x(u_1(s)) \ldots x(u_m(s)))\, ds$$
$$+\int_0^t T(t-s) h(s, x(s), x(v_1(s)), \ldots, x(v_n(s))) ds \qquad (4.4.4)$$

where $LT(t-s)\, g(s, x(s), x(u_1(s)) \ldots x(u_m(s)))$ is integrable for s ϵ (-∞, t] .

4.9.1. EXISTENCE OF SOLUTIONS BY FIXED POINT TECHNIQUE

In this section we establish the existence of mild solutions for neutral systems with multiple delays by the application of Schaefer's Fixed Point Theorem.

Theorem 4.9.1

Consider the system

$$\frac{d}{dt}\left[g(t, x(t), x(u_1(t)), \ldots, x(u_m(t)))\right] = L(t, x_t) + h(t, x(t), x(v_1(t)), \ldots, x(v_n(t))) \qquad \textbf{(4.4.3)}$$

under the following basic assumptions:

(1). For each t ϵ J, the function $h(t, .) : X^{n+1} \to X$, is continuous and for each $(x_0, x_1, \dots, x_n) \epsilon X^{n+1}$, the function h (. $x_0, x_1, \dots, x_n$): [0, τ] → X , is strongly measurable

(2). For each positive number k, there exists a_k ϵ [0 ,τ] such that

$\max_{|\|x_0\|\dots\|x_n\||}$ $h(x_0, x_1, \dots, x_n) \le a_k$, t ϵ J .

(3). The, function g: [0, τ] x X^{m+1} → X is completely continuous and for any bounded set Q_0 ϵ C ((-∞,0] , X) , the set { $g(t, x(t), x(u_1(t)), \dots, x(u_m(t)))$: x ϵ Q_0 } is equicontinuous.

(4). There exist μ ϵ (0,1) and a constant β ,such that $\|(L)\mu\, G(t, x(t))\| \le N_3$, t ϵ J

(5). There exists an integrable function M, M: [0, τ] → [0,∞) such that

$\| h(t, x(t), x(v_1(t)), \dots, x(v_n(t))) \| \le (n+1)m\Omega(\|x\|)$,

where Ω: [0, ∞) → [0, ∞) is a continuous non decreasing function.

(6). $\int_0^{\tau} M(s)ds \le \int_d^{\infty} \frac{ds}{s+\Omega(s)}$

where $d = N_1[\|x_0\| + N_3N_4] + N_3N_4 + \frac{N_3N_2}{\beta}$, $N = \|(L)^{-a}\|$,

$M(t) = N_1 N(t)(n+1)^2$.

Now let us take

$\left(t, x(t), x(u_1(t)), \dots, x(u_m(t))\right) = (t, p(t))$, and

$(t, x(t), x(v_1(t)), \dots, x(v_n(t))) = (t, q(t))$

If the above conditions are satisfied, then the equation (4.4.3) has a mild solution on the interval $[t_0, t_1]$, $t_1 > t_0$.

Proof

We start by converting system (4.4.3) to its' integral equivalent of system (4.4.4) and to operator equation.

Consider the Banach space B = C (J, X) with norm $\|x\| = \sup\{|x(t)| : t \epsilon J\} \le k$.

Define

X (t) = λ (yx)(t) , 0 < λ< 1 .where Ψ: y → y is defined as

$(\Psi x)(t) = T(t)[x(0) + g(0, u(0))] - g(t, w(t)) + \int_0^t LT(t-s)g(s, P(s)ds + \int_0^t T(t-s)h(s, q(s))ds$.

Making use of the basic assumptions of **theorem 4.9.1**, we obtain an estimate of the solution x (t) of equation (4.4.3)

$\|x(t)\| \le N_1[\|x(0)\| + N_3N_4] + [N_3N_4] + N_2 \int_0^t N_3 (t-s)(\beta)^{-1} ds$

$$+ N_1 \int_0^t ((n+1))M(s)\Omega(\ q(s) \ )ds$$

$$\leq N_1[\ \|x(0)\| + N_3N_4] + N_1 \int_0^t (n+1)M(s)\Omega(\ q(s) \ )ds \qquad (4.4.5)$$

Denoting $\|x(t)\|$ in (4.4.5) as less or equal to μ(t) , we have

$\|x(t)\| \leq \mu(t)$, and $\mu(0) = N_1[\ \|x(0)\| + N_3N_4] \ \frac{N_3N_2}{\beta}$

Differentiating μ (t), we have

$$\dot{\mu}(t) = N_1(n+1)M(t)\Omega(\|q(t)\|) \leq \mu(t) [\ \Omega(\mu(t))]$$

This implies $\int_{\mu(0)}^{\mu(t)} \frac{1}{\Omega(s)} ds < \int_0^b M(s)ds \leq \int_{\mu(0)}^{\infty} \frac{ds}{\Omega(s)}$, $0 \leq t < b$ (4.4.6)

The inequality (4.4.6) implies that there is a constant k such that μ (t) ≤ k,

for t ϵ [0 , t_1] and hence , we have

$$\|x(t)\| = \sup \{ |x(t)| : t \in J\} \leq k$$

where k depends only on θ and on the functions M and **Ω**. Evidently the function Ψ is uniformly bounded.

We shall now prove that Ψ: Y → Y is completely continuous operator.

That is, Ψ is relatively compact.

Let Y ={ x ϵ Y : $\|x\| < k$ } for some k > 1 .

We show that Ψ maps B_k into an equicontinuous family.

Let x ϵ B_k and t_1, t_2 ϵ [0 ,b] . Then if $0 \leq t_1 < t_2 < b$, we have

$$(\Psi x)(t_1) = T(t_1)[x(0) + g(0 ,u(0))] - g(t_1,u(t_1)) + \int_0^{t_1} LT(t_1 - s)g(s , u(s))ds$$
$$+ \int_0^{t_1} T(t_1 - s)h(s , v(s)ds \qquad (4.4.7)$$

$$(\Psi x)(t_2) = T(t_2)[x(0) + g(0,u(0))] - g(t_2 ,u(t_2)) + \int_0^{t_2} LT(t_2 - s)g(s , u(s))ds$$
$$+ \int_0^{t_2} T(t_2 - s)h(s , v(s)) ds \qquad (4.4.8).$$

(4.4.7). implies $(\Psi x)(t_2) = T(t_2)[x(0) + g(0,u(0))] - g(t_2 ,u(t_2))$

$$+ \int_0^{t_1} LT(t_2 - s)g(s , u(s))ds + \int_0^{t_1} T(t_2 - s)h(s , v(s))ds$$
$$+ \int_{t_1}^{t_2} LT(t_2 - s)g(s , u(s))ds + \int_{t_1}^{t_2} T(t_2 - s)h(s , v(s))ds .$$

If $0 \leq t_1 < t_2 < b$,then

$\|(\Psi x)(t_1) - (\Psi x)(t_2)\| \leq \|[T(t_1) - T(t_2)][x(0) + g(0 ,u(0)]\|$

$+\|g(t_1 ,u(t_1))-g(t_2,u(t_2))\| + \| \int_0^{t_1} [LT(t_1 - s) - LT(t_2 - s)] [g(s , u(s))] ds \|$

$+ \| \int_0^{t_1} [T(t_1 - s) - T(t_2 - s)][h(s , v(s))]ds \| + \| \int_{t_1}^{t_2} LT(t_2 - s)g(s, u(s))ds \|$

$+\| \int_{t_1}^{t_2} (t_2 - s)h(s , v(s))ds \|$

$\leq \| [T(t_1) - T(t_2)][x(0)+g(0, u(0))] \| + \| g(t_1, u(t_1)) - g(t_2, u(t_2)) \|$

$+ \int_0^{t_1} [LT(t_1 - s) - LT(t_2 - s)] N_3 N_4 \, ds + \int_0^{t_1} [T(t_1 - s) - T(t_2 - s)] \; N_3 N_4 ds$

$+ \int_{t_1}^{t_2} LT(t_2 - s) \; a_k(s) ds \; + \int_{t_1}^{t_2} T(t_2 - s) \; a_k(s) \, ds$ (4.4.9)

The right hand side of (4.4.9) is independent of x є B_k and tends to zero as $t_1 \rightarrow t_2$.

Since Ψ is completely continuous operator T(t) for $t_0 > 0$, implies continuity in the uniform operator topology. Thus Ψ maps B_k onto an equicontinuous family , B_k is uniformly bounded by the estimate of x(t) provided by the system(4.4.9) .

Next, we show that the closure of ΨB_k is compact. Since we have shown that ΨB_k is an equicontinuous family, by Ascoli-Arzelas' theorem, it suffices to show that Ψ maps B_k into a pre-compact set in X.

Let $0 < t < t_1$ be fixed and let ε be a real number satisfying 0 < ε < t for xϵB_k , we define

$(\Psi_\varepsilon x)(t) = T(t)[x(t_0) + g(0, u(0))] - g(t, u(t)) + \int_0^{t-\varepsilon} LT(t-s) g(s, u(s)) ds$

$+ \int_0^{t-\varepsilon} T(t-s) \, h(s, v(s)) \, ds$

$= T(t)[x(t_0) + g(0, u(0))] - g(t, u(t)) - T(\varepsilon) \int_0^{t-\varepsilon} T(-\varepsilon, \varepsilon) h(s, v(s)) ds$.

Since T(t) is a compact operator, the set $Y_\varepsilon(t) = \{ (\Psi x)(t): x \, \epsilon Y_k \}$, t-ε

is pre-compact in X ; for every ε , 0< ε < t .

Moreover, for every x є Y_k , we have

$\| (\Psi x)(t) - (\Psi_\varepsilon x)(t) \| \leq \int_t^{t-\varepsilon} LT(t-s) g(s, u(s)) \; ds + \int_t^{t-\varepsilon} T(t-s) h(s, v(s)) \; ds$

$\leq \int_t^{t-\varepsilon} LT(t-s) \, g(s, u(s)) \; ds + \int_t^{t-\varepsilon} T(t-s) \; a_k(s) \, ds.$

Therefore, there are pre-compact sets arbitrarily close to the set $\{(\Psi x)(t): x \, \epsilon Y_k\}$

Hence the set $\{(\Psi x)(t): x \, \epsilon Y_k\}$ is pre-compact in X.

Ψ: Y → Y is continuous.

Let $\{x_n\}$ be an arbitrary sequence in Y with $x_n \rightarrow x$ in Y,

then there is an integer such that $\| x_n \|$ є Y and x є Y.

By the boundedness of sequence $\{x_n\}$, it follows that

$g(t, u_n(t)) \rightarrow g(t, u(t))$ for each t є J, since

$\| g(t, u_n(t)) - g(t, u(t)) \| \leq 2 a_k(t).$

Invoking the Lebesgue theorem on dominated convergence, we have

$\| \Psi x_n - \Psi x \| = \text{Sup} \| [g(t, u_n(t)) - g(t, u(t))]$

$+ \int_0^t LT(t-s)[g(s, u_n(s)) - g(s, u(s))] ds$

$+ \int_0^t T(t-s)[h(s, v_n(s)) - h(s, v(s))]ds$

$\leq \| [g(t, u_n(t)) - g(t, u(t))] \|$

$+ \int_0^t LT(t-s) \quad [g(s, u_n(s)) - g(s, u(s))] \; ds$

$+ \int_0^t T(t-s) \quad [h(s, v_n(s)) - h(s, v(s))] \; ds.$

Thus Ψ is continuous.

This completes the proof that Ψ is completely continuous.

Finally, the set L(Ψ) given by

$L(\Psi) = \{x \in Y: \; x = \lambda(\Psi x), \lambda \in [0,1]\}$ is bounded.

Consequently, by the application of **Schaefers' fixed point theorem**, the operator Ψ satisfies all the conditions of the theorem and, therefore, has a fixed point in Y.

This fixed point of Ψ is **the mild solution** of the system of interest on J satisfying

$(\Psi x)(t) = x(t).$

CHAPTER 5
SUMMARY, CONCLUSION, RECOMMENDATION AND CONTRIBUTIONS TO KNOWLEDGE

5.1: SUMMARY.

Controllability is one of the fundamental concepts in mathematical control theory. It is a qualitative property of dynamical control systems and is of particular importance to the control theorist. In the recent past, the theory of control of deterministic processes with several degrees of freedom appeared to have reached a satisfying stage of completeness. As interpreted by the theory of nonlinear ordinary differential equations, **Iyai (2006)** the fundamental problems of control theory have been mathematically posed and answered and hence the theory has reached a certain degree of stability and perfection. The authors as a result believed that a thorough and careful presentation of the current status of control theory would serve the useful purpose of offering a foundation on which later researches would be based. It is in this intent, that this work: **"Controllability Results for Nonlinear Neutral Functional Differential Systems"** was carried out**.** Our Objective therefore was to present an organized treatment of control theory that could be complete within the limitations set by the restrictions of deterministic problems identifiable in terms of functional differential equations. Suffice it is to mention here that two kinds of functional differential equations abound:

(a) The Retarded Functional Differential Equation given as

$$\dot{x} = f(t, x_t) \quad ; \quad x(t_0) = \phi = x_{t_0}$$

where ϕ is the initial point and is a function defined in the delay interval [-h,0], h > 0.

(b) The Neutral Functional Differential Equation given as:

$$\frac{d}{dt}\,[D(t, x_t)] = f(t, x_t) \;\; ; \;\; x(t_0) = \phi = x_{t_0}$$

where D is a bounded linear operator

It is easily observed that, both equations are characterized by delays. The motivation for this study stemmed from the fact that most realistic systems should encompass not only the present, but also the past state of the system. This is encountered in many areas of human activities. For a good grasp of the present,(t),some knowledge of the past, (t-h), $t \geq 0$, $h > 0$, is very important. In general, differential equations which include the present as well as the past state of any physical system is called a **Delay Differential Equation (or Functional Differential Equations).**

The Retarded Functional Differential Equations **(RFDE) are** characterized with delays on the state of the system. A typical example is the system

$$\frac{d}{dt}x(t) = x(t-h)\ ,\ \ h > 0$$

$On\ the\ other\ hand, Neutral\ Functional\ Differential\ Equations\ (NFDE)$ are those that have delays on the state as well as in the derivatives. A typical example is the system

$$\frac{d}{dt}\left[x(t) - c\big(x(t-h)\big)\right] = bx(t-h)$$

$$where\ c, b\ , and\ (\ h > 0) are\ constants.$$

Systematic study of controllability started over the years at the beginning of the sixties when the theory of controllability based on the description in the form of state space for both time –varying and time invariant linear control systems was carried out. Roughly speaking, controllability generally means that, it is possible to steer a dynamical control system from an initial state to a final state using the set of admissible controls. Optimal control means doing the same in a best conceivable way. There are many different definitions of controllability which strongly depend on the class of dynamical control systems. In recent years, various controllability problems for different types of nonlinear systems have been considered. However, it should be stressed that, most of the reported work in this direction has been mainly concerned with controllability for linear dimensional systems with constrained control and without delays **(see Klamka (1991), Sun (1996), Underwood and Young (1979)).**Later on delayed differential equations came to limelight **(see Nse (2007), Nse (2007)).**A delayed equation on a linear system is one which affects the evolution of the system in an indirect manner.

If we consider the equation

$$\dot{x} = Ax(t) + Bu(t)$$

where A and B are nxn and mxn matrices. We see that the action of the control is direct in that the local behavior of the trajectory is affected only by the local behavior of the control u(t) at time t. It is known that, most natural applications give rise to mechanism of indirect actions where decisions in the control function are shifted, twisted or combined before affecting the evolution thus comprising the delay u(t-h) represented by the system

$$\dot{x} = Ax(t) + Bu(t-h)$$

It is well known that, the future state of realistic models in the Natural Science, Economics and Engineering depends not only on the present, but also on the past state and at times , even on the derivative of the past state. There are simple examples from Biology (predator-prey, Lodka Volterra, Spread of Epidemics), from Economics (dynamics of capital growth of global economies) and from Engineering (mechanical and aero – space , aircraft stability, automatic steering using minimum fuel and effort, control of high speed, closed air circuit, etc, this research effort are therefore intended to forge far-reaching solutions to these daily human endeavors.

Objective.

Our principal objective in this work was to obtain **Necessary and Sufficient Conditions** for controllability, optimal control and stability for Neutral Functional Differential Systems. It is known from **Onwuatu (1993)** that, if a system can be shown to be relatively controllable, then optimal control is unique and Bang-Bang. In the light of this, we have considered the Neutral Volterra Integrodifferential Equation of the form

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t,s)x(s)ds - g(t)\right]$$

$$= A(t)x(t) + \int_0^t G(t,s)x(s)ds + \int_{-h}^0 d_\theta H(t,\theta)u(t+\theta)$$

with initial condition $x(t_0) = x_0$,where $x \epsilon E^n$ is the state space and $u \epsilon E^m$ is the control function, $H(t,\theta)$ is an nxm matrix continuous at t and of bounded variation in θ on $[-h,0]$; $h>0$ for each $t \epsilon [t_0,t_1]$; $t_1 > t_0$ .The nxn matrices $A(t)$,$C(t,s)$,$G(t,s)$ are continuous in their arguments. The n-vector function g is absolutely continuous.

The above system was investigated for existence, form and uniqueness of optimal control by first of all considering the relative controllability of the system.

Scope.

Differential systems are generally important tools for harnessing different components into a single system and analyzing the inter-relationships that exists between them which otherwise might continue to remain independent of each other. Physical systems which express the present state of solutions are the most common system encountered in the theory of differential equations. The Scope of this work therefore went beyond these systems and addressed more realistic systems involving not only the present but also the past states of the system. This is because the latter permeates various aspects of life and has of late triggered interest in research. Neutral differential equations arise in many areas of applied Mathematics and such equations have received much attention in recent years. For example, the mixed initial boundary hyperbolic partial differential equations which arise in the study of lossless transmission lines can be replaced by an associated neutral differential equation. This equivalence has been the basis of a number of investigations of the stability properties of distributed networks**, (see Iyai (2006), Kwun (1991)).**

It is in this light also that we intended to broaden the scope to involve systems of the Neutral type. This is because in recent years, it has emerged as independent branch of modern research due to its connection to many fields such as continuum mechanics, population dynamics, system theory, biology epidemics and chemical oscillations **(see Balachandran, Balasubramaniam and Dauer (1996), Burton (1983), Corduneanu (1985)).**

In our system, we studied the linear neutral volterra integro-differential equation. The purpose of our investigation was firstly to obtain necessary and sufficient conditions for the relative controllability of the equation. Secondly, we went beyond the controllability of the system to obtaining the existence, form and uniqueness of the optimal control. Volterra-integro-differential equation has wide application in applied mathematics and engineering; this underscores its importance in mathematical control theory.

We obtained a variation of constant formula for the solution of the equation and were able to extract the reachable set, the attainable set and the controllability grammian. These are the ingredients needed for the controllability results. In **theorem 4.1**, it was established that the system is relatively controllable if the controllability grammian is non-singular. This result made it possible for the invertibility of the grammian and the use of it in forming a control which can steer the initial state to the final state. It was evident from the theorem, that relative controllability shares the same dimension of meaning with properness of the system.

Theorem 4.2 showed that the system is relatively controllable if and only if 0 is in the interior of the reachable set supports the closeness and convexity of the reachable set which in turn supports its normality. A system is normal when the reachable set is strictly convex. This gives form to the optimal control. By reversed argument, if 0 is not in the interior of the reachable set, then it is on the boundary and optimal control is achieved at the first contact of the moving or stationary target with the attainable set.

This argument is supported by **theorem 4.1** to establish the existence of optimal control. The uniqueness and form of control is derived from calculus of variation based on the maximization of the objective function y(t, u) in the reachable set.

We have then forged ahead to achieve solution near the origin to another Neutral Functional Differential Equation of the form

$$\frac{d}{dt}[D(t,x_t)] = L(t,x_t)x_t + f(t,x_t) + \int_{-\infty}^{0} A(t)x(t+\theta)d\theta$$

$where$,

$$L(t,x_t) = \sum_{k=0}^{\infty} A_k x(t-w_k) + \int_{-\infty}^{0} A(t,\theta)x(t+\theta)d\theta$$

is a bounded linear operator, and f(t,x_t) is a perturbation function. The nxn matrix functions A_k and A(t ,θ) are measurable in (t, s) ϵ ExE, θ ϵ [-∞,0).

The energy method of Alexander Mikhailovick Lyapunov **(1829)** which stipulates that, in a stable system, the total energy in the system will be a minimum at the equilibrium point was used to establish results. This no doubt paved the way for discussions on stability of various nonlinear functional equations.

The statements of the problem were thus formulated:

Suppose we are given a Neutral Functional Differential Equation as in equations above, and it is required to move the solution $\boldsymbol{x(t)}$ from an initial point $\boldsymbol{x_0}$ at time $\boldsymbol{t_0}$ to a terminal point $\boldsymbol{x_1}$ at time $\boldsymbol{t_1}$.The problem arises as to whether it is possible to carry out this task in finite time. As an illustration, we considered the system

$$\dot{x} = -ax(t)$$

Clearly, the solution of the above system is

$$x(t) = Ae^{-at}$$

If we desire to drive this solution to the origin .that is, null controllability, we observed that, it cannot be achieved in **finite time** because $x(t)$ tends to zero only when t tends to infinity. Since this cannot be achieved in finite time, we need to modify the system to be able to bring $\boldsymbol{x(t)}$ to 0 in **finite time.** The process of modification is called controllability which answered the controllability problem. The optimal control problem is formulated as follows: Having guaranteed controllability of the system in question is there an admissible control **u*** such that the solution $\boldsymbol{x(t, \phi, u^*)}$ of the system hits a continuously moving target point in minimum time **t*.** Here u* is the **optimal control** and t* the **minimum time.** Once it is guaranteed that such a control exists, we showed it is unique and Bang-Bang.

Finally, the system

$$\frac{d}{dt}[D(t,x_t)] = L(t,x_t)x_t + f(t,x_t) + \int_{-\infty}^{0} A(t,\theta)x(t+\theta)d_\theta : x_{t_0} = \phi.$$

(a nonlinear infinite neutral system)

was presented and investigated for global uniform asymptotic stability in the large

We considered the system below,

$\frac{d}{dt}[D(t,x_t)] = L(t,x_t)x_t + f(t,x_t) + \int_0^\infty A(t,\theta)x(t+\theta)d_\theta$

(circularity of the function from -∞ to 0, and from 0 to ∞).

We linearized the system as in **Chukwu (1992)** by setting x_t = z in L; a specified function inside the function $L(t,x_t)x_t$ to have L(t, z) x_t with no loss of generality.

Thus the system becomed

$$\frac{d}{dt}[D(t,x_t)] = L(t,z)x_t + \int_0^\infty A(t,\theta)\, x(t+\theta)d_\theta + f(t,x_t)$$

Evidently

$$L(t,z)x_t = \sum_{k=0}^{\infty} A_k x(t-w_k) + \int_{-\infty}^{0} A(t,\theta)\, x(t+\theta)d_\theta + \int_0^\infty A(t)\, x(t+\theta)d_\theta$$

$$L^*(t,z)x_t = \sum_{k=0}^{\infty} A_k x(t-w_k) + \int_{-\infty}^{\infty} \mathrm{A}(\mathrm{t},\theta)\, \mathrm{x}(\mathrm{t},\theta)\mathrm{d}_\theta$$

The representationsL, L* are same under the following assumptions

$$L(t,z)x_t = \lim_{p\to\infty} \sum_{k=0}^{p} A_k x(t-w_k) + \lim_{M,N\to\infty} \int_M^N \mathrm{A}(\mathrm{t},\theta)\, \mathrm{x}(\mathrm{t}+\theta)\mathrm{ds}$$

We assume the limits exist, giving finite partial sum for the infinite series and the improper integrals. Thus the system

$$L^*(t,z)x_t = \sum_{k=0}^{\infty} A_k x(t-w_k) + \int_{-\infty}^{\infty} \mathrm{A}(\mathrm{t},\theta)\, \mathrm{x}(\mathrm{t}+\theta)\mathrm{d}_\theta$$

is finite and well defined function.

In the light of the above, the system reduced to

$$\frac{d}{dt}[\mathrm{D}(\mathrm{t},\mathrm{z})x_t] = \mathrm{L}(\mathrm{t},\mathrm{z})x_t + f(t,x_t);\; x(t_0) = \phi \in \mathrm{C}$$

where $\mathrm{L}(\mathrm{t},\mathrm{z})x_t = \sum_{k=0}^{p} \mathrm{A_k}\, \mathrm{x}(\mathrm{t}-\mathrm{w_k}) + \int_{-h}^{0} \mathrm{A}(\mathrm{t},\theta)\, \mathrm{x}(\mathrm{t}+\theta)\mathrm{d}\theta$

Integrating the system, after linearizing, we have the solution

$$x(t) = x(t,t_0,\phi,0) + \int_0^t \mathrm{X}(\mathrm{t},\mathrm{s})\, \mathrm{f}(\mathrm{s},x_s)\mathrm{ds}$$

where X(t ,s) is the fundamental matrix of the homogenous part of the system .

X (t , s) = I (identity matrix); t = s

we asked the question: Is the solution near the origin of system (1.1.8) going to remain quite close for all future times? This is the stability problem which we are desirously answered in the affirmative.Theorem4.8 established the conditions for exponential asymptotic stability in the large of our nonlinear neutral system(4.2.2). From the proof of theoem4.8, we observed that system(4.2.2) vis-à-vis system(4.2.9) is uniformly exponentially stable in the large implies that the solution of the large implies the solution of the system has an exponential estimate. That is $\| x_t(t_0,\phi) \| \le \beta\, e^{c(t-t_0)} \|\phi\|$.

Clearly, as $t \to \infty$, $\| x_t\ (t_0,\ \phi) \| \to 0$ Since the trivial solution has been shown to be uniformly stable, system (4.2.9) is said to be uniformly asymptotically stable. Evidently, uniform exponential asymptotically stable in the large of a system guarantees its uniform asymptotic stability. We stated the realization as a theorem4.9,thus.

Theorem 4.9: (Global Uniform Asymptotic Stability).

Consider the nonlinear neutral system (4.2.9)

$$\frac{d}{dt}[\ D(t, x_t)\ ] = L(t\ , z)\ x_t + f(t\ , x_t) + \int_{-\infty}^{0} A(t)\ x(t+\theta)d\theta\ \ ;\ \ x_{t_0} = \phi$$

Under all the assumptions of theorem 4.8, assume further that

System (4.2.9) satisfies $f(\ t\ , 0) = 0$,

then system (4.2.9) is uniformly exponentially asymptotically stable.

To extend the existence results for non-linear abstract neutral differential equations in the work of **Balachandran, Leelamani and Kim (2006)** to the system with delay base was the objective of our research here. We therefore, presented the system

$$\frac{d}{dt}\left[x(t) + g(t, x(t), x\big(u_1(t)\big), \dots, x(u_m(t))\right] = L(t, x_t) + h(t, x(t), x\big(v_1(t)\big), \dots, x\big(v_n(t)\big))$$

$x\ (\theta) = 0\ ;\ \theta \in [-h, 0],\ t \in J = [0, t_1],\ t_1 > t_0$,

with the purpose of obtaining **mild solutions** of the system in the Banach Spaces, using **Schaefer's fixed point theorem**.Theorem4.9.1 established the conditions for the existence of mild solution of the system on the interval $J = [0,t_1]$, $t_1 > t_0$.

5.2: CONCLUSION

In this work, the linear neutral volterra-integro-differential equation was presented for study. Sufficient conditions for its relative controllability, existence, form and uniqueness of optimal control were also derived. It was proved that the optimal control for the system exists if only the system is relatively controllable on finite interval. The form of the optimal control is as given in equation (4.1.6). The establishment of its uniqueness provided a new approach. The optimal control is Bang-Bang and the immediacy of the applicability of the Bang-Bang Principle is guaranteed with the state space being the Euclidean space. **(See Corduneanu and Lakshmikatham (1980)).**

A nonlinear infinite neutral system was presented for stability problem. Necessary and sufficient conditions for solutions of the system near the origin of the system being quite close for all future times were derived. Theorem4.8 and Theorem4.9 established the conditions for exponential asymptotic stability in the large of our nonlinear neutral system(4.2.2) and global uniform asymptotic stability respectively.

The existence results for non-linear abstract neutral differential equations in the work of **Balachandran, Leelamani and Kim (2006) was extended to** the systems with delay base of the form

$$\frac{d}{dt}\left[x(t) + g(t, x(t), x\big(u_1(t)\big), \dots, x(u_m(t))\right] = L(t, x_t) + h(t, x(t), x\big(v_1(t)\big), \dots, x\big(v_n(t)\big))$$

x (θ) = 0 ; θ ϵ [-h, 0], t ϵ J = [0, t_1], $t_1 > t_0$

with the purpose of obtaining **mild solutions** of the system in the Banach Spaces, using **Schaefer's fixed point theorem**.

Theorem4.9.1 established the necessary and sufficient conditions for the system to have a mild solution on the interval $[t_0, t_1\ ]$.

5.3: RECOMMENDATION

We recommend that the easiest way to show that a dynamical control system is null controllable is to show that the system is relatively controllable in a finite interval.

We recommend the new approach to investigating the existence of an optimal control in a system. The new approach is to test for the uniqueness of an admissible control of the given system .This is because; we have established that, once an admissible control is proved to be unique, then it is an optimal control and Bang-Bang.

The methods and arguments used in this work can be applied to systems of semi-linear neutral systems of the Volterra type and abstract neutral differential systems.

5.4: CONTRIBUTIONS TO KNOWLEDGE

We have made the following contributions to knowledge in this work:

We have established necessary and sufficient conditions for the controllability, existence, form and uniqueness of optimal control of Neutral Volterra Integrodifferential Systems, using **relative controllability equivalence**.

We have established necessary and sufficient conditions for the global uniform asymptotic stability of Nonlinear Infinite Neutral Functional Differential Systems, using the **energy method** of Alexander Mikhailovick Lyapunov (1829) which stipulates that ,in a stable system, the total energy in the system will be a minimum at the equilibrium point.

We have extended the existence results for Nonlinear Abstract Neutral Differential Systems to the systems with delay base, using **Schaefers' fixed point theore**

REFERENCES

1 Ananjevskii I.M and V.B. Kolmanovskii (1990), "Stabilization of Some Nonlinear Hereditary Mechanical Systems", Nonlinear Analysis, Theory, Methods Applications, no 2,pp 101-114.

2 Angell, T.S(1990), "On Controllability for Non Linear Hereditary Systems, a Fixed Point Approach," Nonlinear Analysis, Theory Methods and Applications 4 , pp529-548.

3 Artstein .Z(1982), "Linear Systems with Delayed Control- A Reduction", I.E.E.E. Transactions on Automatic Control AC- 27, pp869- 879.

4 Artstein .Z. and Tradmar. G.(1982), "Linear Systems with Indirect Controls – the Underlying Measure", SIAM Journal on Control and Optimization 20 pp 96-111.

5 Asuquo.B and E. Usah (2008)"Control of Linear Systems Using Pure Time Delay", Journal of NAMP Vol. 12, pp55-62.

6 Balachandran .K. and Dauer, J.P(1989) "Relative Controllability of Perturbation of Nonlinear Systems", Journal of Optimization Theory and Applications 63 ,pp 51-56.

7 Balachandran, K.,P. Balasubramaniam and J.P. Dauer(1995), :Controllability of Quasilinear Delay Systems in Banach Spaces," Optim. Control Applications; Methods 16 ,pp 283-290.

8 Balachandran .K. and Dauer, P.J. (1996), "Relative Controllability of Nonlinear Neutral Volterra Integrodifferential Systems", Australian Mathematics Society.

9 Balachandran .K. and Dauer, P.J(1996) ,Local Null Controllability of Nonlinear Functional Differential Systems in Banach Spaces, Journal of Optimal Theory and App. 88, pp61-75.

10 Balachandran, K; Balasubramaniam .P. and J.P Dauer(1996), "Local Null Controllability of Nonlinear Functional Differential Systems in Banach Spaces", Journal of Optimal Theory and Application 88, pp 61-75

11 Balachandran . K, J.P. Dauer & P Balasubramaniam(1997) Asymptotic Neutral Volterra Integrodifferential Systems. Journal of Mathematical systems Estimation vol. 7 No. 2 pp 1-4.

12 Balachndran, K; Balasubramaniam .P. and J.P Dauer(1997), "Asymptotic Null Controllability of Nonlinear Neutral Volterra Integrodifferential Systems", Journal of Mathematical System vol .7 No.2 pp1-14 .

13 Balachandran . K, J.P. Dauer (1998), Existence of Solutions of a Nonlinear Mixed Neutral Equations,Applied Mathematics Letters11 No2,pp23-28.

14 Balachandran. K. and R. Sakthivel(1999), Existence of Solutions of Neutral Functional Integrodifferential Equations in Banach Spaces Proceeding of the Indian Academic Sciences 109, pp325-332

15 Balachandran .K. and J.P. Dauer (2002), Controllability of Nonlinear Systems in Banach Spaces A Survey Jota 115.

16 Balachandran and E.R. Anandhi,(2003) "Neutral Functional Integrodifferential Control Systems in Banach Spaces", Kybernetika 39 No 3,PP359-367.

17 Balachandran and A. Leelamani (2006) "Null Controllability of Neutral Evolution Integrodifferential Systems with Infinite Delay", Hindawi Publishing Cooperation Mathematical Problems in Engineering, .

18 Balachandran K, J. Kim, A.Leelamani(2006): Existence Results for NonlinearAbstraNeutral Differential Equations with Time Varying Delays. Applied Mathematic-Notes 6, pp186-193.

19 Burton, T.A(1983), "Volterra Integral and Differential Equations", Academic Press, NewYork.

20 Cheban,E.N (2000), Uniform Exponential Stability Linear Almost Periodic Systems in Banach Spaces, Electronic Journal of Differential Equations ,Vol.2 N029, pp1-18

21 Chukwu, E.N (1981); Global Asymptotic Behaviour of Functional Differential Equations of the Neutral Type JOTA, vol.5, pp853-872.

22 Chukwu E.N. (1982): The Time Optimal Control Problem of Linear Neutral functional differential Systems,Journal of Nigerian Mathematical society .pp334-335

23 Chukwu, .E.N (1986), Compactness of Core of Targets for Delays Systems, Journal of Optimization Theory and Application, Vol.14.

24 Chukwu, E.N (1987), On the Endidean Controllability of Neutral Systems with Nonlinear Base, Nonlinear Analysis, Theory, Methods and Applications 11 pp 115-123.

25 Chukwu E.N.(1988): The Time Optimal Control Theory of Linear Differential Equations of Neutral Type. Journal Computer. Math & Applications Vol. 16, pp851, 866.

26 Chukwu, E.N (1992), Stability and Time Optimal Control of Hereditary Systems, Academic Press New York.

27 Chukwu, E.N (2001), Differential Models and Neutral Systems for Controlling the Wealth of Nations, Services on Advances in Mathematics fro Applied Sciences, Vol. 54. World Scientific, New Jersey.

28 Conner, D.A.O (1978), State Controllability and Observability for Linear Neutral Systems", D.SC. Thesis Washington University.

29 Corduneanu C (1985), Integral Equations and Applications, Cambridge University Press Cambridge.

30 Coreduneanu .C. and V. Lakshmikatham (1980), "Equations with Unbounded Delay", a Survey, Nonlinear Analysis, Theory, Methods & Applications 4 ,No 5, pp831-877.

31 Cruz,M.A and Hale,J.k (1970), Stability of Functional Differential Equations vol.7 pp334-335

32 Dacka. D (1998), On the Controllability of a Class of Nonlinear Systems, I.E.E.E., Trans Automatic Control AC-25 , pp263-266.

33 Eke, A.N (1990); Total Controllability for Nonlinear Perturbed Systems Jour Inst. Math and Computer Science Vol. 3 No.3,pp 335-340.

34 Eke, A.N (2000); Stabilizability for Linear Feedback Observable Systems; Journal of the Nigerian Mathematical Society Vol. 10 pp59-68 .

35 Fu, X.L92003), Controllability of Neutral Functional Differential Systems in Abstract Spaces", Applied Mathematics and Computation 141, No 2-3,pp 281-296.

36 Fu; X.L., and K. Ezzinbi (2003), "Existence of Solutions for Neutral Functional Differential Evolution Equations with Non Local Conditions, Nonlinear Analysis 54 , No 2, pp 215-227.

37 Fu (2004), Controllability of Abstract Neutral Functional Differential Systems with Unbounded Delay, Applied Mathematics and Computation 151, No 2, pp299-314.

38 Gahl, R.D(1978), Controllability of Nonlinear Systems of Neutral Type, Journal of Mathematical Analysis and Applications 63 ,pp 33-42.

39 Gyori. I. and Wu. J (1991). A Neutral Equation Arising from Compartmental Systems With Pipes, Journal of Dynamics and Differential Equations 3 ,pp 289-311.

40 Hajek,O (1974): Core of Targets in Linear Control System, Maths .System Theory, Springer Verlag, New York,Vol.8 No.3,pp203-206.

41 Hale J. (1977) Theory of Differential Equations, Applied Math, Science Vol3 Springer Verlag, New York.

42 Hmanmed A (1986), Stability Conditions of Delay Differential Systems, Int .Journal of Control vol.43, No2, pp455-463.

43 Iheagwam, V.A. (2002); Characterization of Cores of Targets for Systems with Distributed Delays in Control, Journal of Nigerian Mathematical Society Vol.21 .

44 Iheagwam V.A and J.U Onwuatu (2005), Relative Controllability and Null Controllability of Linear Delay Systems with Distributed Delays in the State and Control,Journal of Nigerian Association of Math Physics Vol. 9 , pp 221 -238.

45 Iheagwam, V.A.and Nse C.A (2007): On the Controllability of Economic Growth of third World Countries; Journal of Mathematical Sciences, Vol. 1, pp31-37.

46 Iyai D (2006): Euclidean Null Controllability of Infinite Neutral Differential Systems, Anziam Journal, 48, ,48,pp285-293 .

47 Klamka, J. (1976), Relative Controllability of Nonlinear Systems With Distributed Delays in Control, International Journal of control 28 ,pp 307-312.

48 Klamka, J. (1976), Relative Controllability and Minimum Energy Control of Linear Systems with Distributed Delay in Control, E.E.E. Trans Automatic Control AC-21.

49 Klamka Jerzy (1980), Controllability of Nonlinear Systems with Distributed Delay in Control, Int. Journal, Control , Vol;31, No 5,pp 811-819.

50 Klamka, J. (1991); Control of Dynamical Systems, Dondiecht Wawer Academic Publishers.

51 Klamka.J.(2004); Constrained Controllability of Semi-linear Systems with multiple delays in Control, Bulletin of the Polish Academy of Sciences,Vol.82 No1

52 Kwun, J.Y., Park and J.W. (1991), Ryu, Approximate Controllability and Controllability for Delay Volterra Systems, Bullkorean Math. Soc. 28

53 Nse, C.A (2007): Constrained Controllability of Infinite Dimensional Systems with Single Point Delay in the Control. Research Journal of Applied Sciences,2(7),pp804-806 .

54 Nse, C.A (2007): Constrained Relative Controllability of Semilinear Dynamical Systems with Multiple Delays in the State and Control, Journal of the Nigerian Association of Mathematical Physics,Vol11.pp275-278.

55 Onwuatu, J.U. (1984); On the Null Controllability in Function Space Nonlinear Systems of Neutral Functional Differential Equations with Limited Controls, Journal of Optimization Theory and Applications 42 ,pp 397-420.

56 Onwuatu, J.U. (1990), On Controllability and Null Controllability of Linear Systems With Distributed Delays in the Control", Journal of Differential and Int. Equations.

57 Onwuatu, J.U (1993): Null Controllability of Nonlinear Infinite Neutral Systems, Kybernetika, vol.29 , pp1-12.

58 Onwuatu, J.U (2002): On Criteria for Closeness of an Attainable Set of a Discrete Nonlinear Control Systems, Journal of Mathematical Analysis and Applications 268,pp 404-497.

59 Olbrot, A.W (1972): On Controllability of Linear Systems with Time delay in Control: I.E.E.E. trans. on Automatic Control ac-16, pp664-666.

60 Robertson E.F and Oconnor T.J (2005);Biography of Vito Volterra,(3/5/1860-11/10/1940), publication of the school of Mathematics and Statistics, University of St. Andrew Scotland.

61 Sebakhy, O AND Bayoumi, M.N (1973); Controllability of Linear Time Varing Systems with Delay in the Control; Int. J. on Control .

62 Sontag, E.D: Kalmans (1994); Controllability Rank Conditions, from Linear to Nonlinear (Pre-print).

63 Sun, N.K:A (1996); Unified Approach for Constrained Approximate Controllability for the Heat Equation and retarded equations, Journal of Mathematical Analysis and Applications vol150 pp1-19.

64 Umana R.A. (2008), Relative Controllability of Neutral Functional Differential Systems with Nonlinear base, Journal of NAMP. Vol. 13 , pp 261-266.

65 Underwood, R.G and Young, D.F (1979): Null Controllability of Nonlinear Functional Differential Equations, SIAM Journal of Control and Optimization, Vol .7, pp763- 772.

66 Underwood and Chukwu, E.N. (1988); Null Controllability of Nonlinear Neutral Differential Equations, Journal of Mathematical Analysis and Applications, 29, pp326- 345.

67 Wei, K.C (1976): I.E.E.E Trans. Autom Control 21, pp787.

Appendix A

3.10. Existence Theorems

Theorem 3.10.1. (Schauder's Fixed Point Theorem)

If A is a closed, bounded and convex subset of a Banach space B and if the map T: A → A is completely continuous, then there is a point $z \epsilon A$ such that T(z) = z, that is ; z is a fixed point

Theorem 3.10.2 (Schaefer's fixed point theorem),

Let B be a normed linear space. Let **g:** B → B be completely continuous operator, that is, it is continuous and image of any bounded subset is contained in a compact set, and let **Qg** = {x ϵ B: x = λ**g**x ,for λ ϵ (0, 1), that is, 0 < λ < 1},then either **Qg** is unbounded or **g** has a fixed point.

Definition 1.12.1: (mild solution)

A function $x(.)$ is called **a mild solution** of the system (1.5.2) if

$x(t) = 0$, for t ϵ (-∞ ,0] ,

the restriction of $x(t)$ to the interval [0, τ] is continuous and for each [0, τ], the function $x(t)$ satisfies system(4.4.3). That is, the function $x(t)$ satisfies the following integral equation:

$$x(t) = T(t)\{0 + g[0, x(u_1(0)),\ldots,x(u_m(0))] - g\{t, x(t), x(u_1(t)),\ldots,x(u_m(t))\}$$
$$- \int_0^t LT(t-s)\, g(s, x(s), x(u_1(s)) \ldots x(u_m(s)))\, ds$$
$$+ \int_0^t T(t-s) h(s, x(s), x(v_1(s)), \ldots, x(v_n(s))) ds \quad (4.4.3)$$

where $LT(t-s)\, g(s, x(s), x(u_1(s)) \ldots x(u_m(s)))$ is integrable for s ϵ (-∞, t] .